REDUCED TWO-TYPE DECOMPOSABLE CRITICAL BRANCHING PROCESSES WITH POSSIBLY INFINITE VARIANCE

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Abstract. We consider a Galton-Watson process $\mathbf{Z}(n) = (Z_1(n), Z_2(n))$ with two types of particles. Particles of type 2 may produce offspring of both types while particles of type 1 may produce particles of their own type only. Let $Z_i(m,n)$ be the number of particles of type i at time $m < n$ having offspring at time n. Assuming that the process is critical and that the variance of the offspring distribution may be infinite we describe the asymptotic behavior, as $m, n \to \infty$ of the law of $\mathbf{Z}(m,n) = (Z_1(m,n), Z_2(m,n))$ given $\mathbf{Z}(n) \neq \mathbf{0}$. We find three different types of coexistence of particles of both types. Besides, we describe, in the three cases, the distributions of the birth time and the type of the most recent common ancestor of individuals alive at time $n \to \infty$.

1. Introduction

Describing the genealogy of populations yields a better understanding of patterns of variation in genetic data. It is a key issue in population genetics and has been the subject of many investigations in recent years. In this work, we are interested in the genealogy of a population modeled by a two-type decomposable Galton-Watson branching process conditionally on its survival until a large time n.

Let $Z_i(n)$ be the number of type i particles ($i \in \{1, 2\}$) alive at time $n \in \{0, 1, ...\}$, $\mathbf{Z}(n) = (Z_1(n), Z_2(n))$, and denote by $\mathbf{Z}(0) = \mathbf{e}_i$ the event

$$Z_1(0) + Z_2(0) = Z_i(0) = 1.$$

We assume that the process is critical, that is to say

$$\mathbf{E}\left[Z_i(1)|\mathbf{Z}(0) = \mathbf{e}_i\right] = 1, \quad i \in \{1, 2\}.$$

Type 1 particles may produce only particles of their own type while type 2 particles are able to produce particles of both types. The production of type 1 particles by type 2 particles can be seen as recurrent mutations or migrations. In the finite variance case, this process has been studied in [15, 19], and in the case of infinite variance it has been investigated in [28] and [24] in detail. Zubkov [28] and Vatutin and Sagitov [24] have analyzed the asymptotic behavior of the survival probability of the process given $\mathbf{Z}(0) = \mathbf{e}_2$, and described the conditional limiting distribution of the population size at time $n \to \infty$ given $\mathbf{Z}(n) \neq \mathbf{0}$ (where $\mathbf{0} = (0, 0)$) and $\mathbf{Z}(0) = \mathbf{e}_2$. Our motivation for considering the infinite variance case comes from [6]. In this work Eldon and Wakeley, using genetic data from a population of pacific oysters, showed that one individual was able to generate enough offsprings in one birth event to produce a significant proportion of the population. This is also the case for many types of fungi and viruses.

The present paper deals with the structure of the family tree of the process $\{\mathbf{Z}(m), 0 \leq m \leq n\}$ given $\mathbf{Z}(n) \neq \mathbf{0}$. More precisely, we investigate the properties of the process

$$\mathbf{Z}\left(m, n\right) = (Z_1(m, n), Z_2(m, n))$$

where $Z_i\left(m, n\right)$ is the number of type i particles alive at time $m < n$ and having descendants at time n. The process $\mathbf{Z}\left(\cdot, n\right)$ is called a reduced branching process and can be thought of as

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the family tree relating the individuals alive at time n. Two important characteristics of the reduced process are the birth moment β_n and the type $\mathcal{T}_n$ of the most recent common ancestor (MRCA) of all individuals alive at time n defined as

$$\beta_n = \max\left\{m < n : Z_1(m,n) + Z_2(m,n) = 1\right\}, \quad \mathcal{T}_n = \{i \in \{1,2\} : Z_i(\beta_n, n) = 1\}. \tag{1}$$

Limiting reduced trees have been extensively studied in the literature. In the monotype and finite variance case, the distribution of β_n already appears in Zubkov [27], and the full structure of the reduced tree is derived by Fleischmann and Siegmund-Schultze [8]. Analogous results in the stable case and/or in the more general setting of multitype indecomposable branching processes with finite or infinite offspring variance can be found in Vatutin [16, 20] and Yakymiv [26]. Finally, Duquesne and Le Gall [5] derived general properties of limiting reduced trees in the continuous setting.

For $s_1, s_2 \in [0,1]$, let

$$F_1(s_1) = \mathbf{E}\left[s_1^{Z_1(1)} | \mathbf{Z}(0) = \mathbf{e}_1\right] \quad \text{and} \quad F_{21}(s_1,s_2) = \mathbf{E}\left[s_1^{Z_1(1)} s_2^{Z_2(1)} | \mathbf{Z}(0) = \mathbf{e}_2\right]$$

be the offspring generating functions for particles of both types. Our basic assumptions on the offspring distributions are formulated in terms of the generating functions $F_1(s_1)$ and $F_{21}(s_1,s_2)$ as $s_1, s_2 \uparrow 1$:

$$F_1(s_1) = s_1 + (1-s_1)^{1+\alpha_1} L_1(1-s_1), \tag{2}$$

and

$$F_{21}(s_1,s_2) = s_2 + (1-s_2)^{1+\alpha_2} L_2(1-s_2) - (A_{21} - \rho(s_1,s_2))(1-s_1), \tag{3}$$

where $0 < \alpha_1, \alpha_2 \le 1$, $L_1(x)$ and $L_2(x)$ are slowly varying functions as $x \downarrow 0$,

$$A_{21} = \mathbf{E}\left[Z_1(1) | \mathbf{Z}(0) = \mathbf{e}_2\right] \in (0, \infty), \tag{4}$$

and the function $\rho(s_1,s_2) \to 0$ as $s_1 s_2 \uparrow 1$. The form of the probability generating function F_1 is natural if we aim at modeling a population where the offspring distribution has an infinite variance. Indeed, (2) holds if and only if the distribution of $Z_1(1)$ is in the domain of attraction of a stable law with index $1+\alpha_1$. Such a monotype branching process has been investigated by many authors (see for example [13, 4, 25, 10]).

Introduce the following notation, for $i \in \{1,2\}$:

$$Q_i(n) = \mathbf{P}\left(Z_i(n) > 0 | \mathbf{Z}(0) = \mathbf{e}_i\right) \quad \text{and} \quad Q_{21}(n) = \mathbf{P}\left(\mathbf{Z}(n) \neq \mathbf{0} | \mathbf{Z}(0) = \mathbf{e}_2\right).$$

It is known [13] that, given Condition (3) with $s_1 = 1$ and Condition (2),

$$Q_i(n) = (n+1)^{-1/\alpha_i} l_i(n), \quad n \to \infty, \tag{5}$$

where $l_i(n)$ is a slowly varying function as $n \to \infty$,

$$\alpha_i n Q_i^{\alpha_i}(n) L_i(Q_i(n)) \sim 1, \quad n \to \infty, \tag{6}$$

and for $\lambda > 0$,

$$\lim_{n\to\infty} \mathbf{E}\left[e^{-\lambda Q_i(n) Z_i(n)} | \mathbf{Z}(0) = \mathbf{e}_i, Z_i(n) > 0\right] = 1 - (1 + \lambda^{-\alpha_i})^{-1/\alpha_i}. \tag{7}$$

We will see that three types of behavior emerge for the reduced trees, depending on the relative values of α_1 and α_2. More precisely, we will distinguish three cases:

$$nQ_1(n) = o(Q_2(n)), \ n \to \infty, \quad \text{(Theorems 1 and 4)}, \tag{8}$$

$$Q_2(n) = o(nQ_1(n)), \ n \to \infty, \quad \text{(Theorems 2 and 5)}, \tag{9}$$

$$nQ_1(n) \sim \sigma Q_2(n), \ \sigma \in (0,\infty), \ n \to \infty, \quad \text{(Theorems 3 and 6)}. \tag{10}$$

We will show that under Condition (9), the reduced processes have similar properties as in the case of finite variance studied in detail by Vatutin in [20] and [23], while under condition (8) or (10) the dynamics of the reduced processes has an essentially different nature.

The paper is organized as follows. In Section 2 we describe the local behavior of the reduced process. Section 3 is devoted to its global behavior and to the law of the MRCA of individuals alive at a large time n. In Section 4 we give auxiliary results needed in the proofs. Sections 5 to 7 are devoted to the analysis of the local structure of the reduced process. In Sections 8 and 9 we consider the global behavior of the reduced process. Finally, in Section 10, we derive the MRCA limit distributions.

In the sequel $\mathbf{P}_i$ and $\mathbf{E}_i$ will denote the probability and expectation conditionally on $\mathbf{Z}(0) = \mathbf{e}_i$. Sometimes it will be convenient to write $\mathbf{P}$ and $\mathbf{E}$ for $\mathbf{P}_2$ and $\mathbf{E}_2$. Finally, the relations

$$a(n) \sim b(n), \quad a(n) = O(b(n)), \quad a(n) = o(b(n)), \quad \text{and} \quad a(n) \ll b(n)$$

will be understood (if otherwise is not stated) as $n \to \infty$ and the symbols $C, C_1, C_2, ...$ will denote positive constants which may vary from line to line.

2. Local characteristics of the limit processes

We first deal with Condition (8). Notice that α_1 is necessarily less than 1, as otherwise $nQ_1(n)$ would be a slowly varying function while $Q_2(n)$ is a regularly varying function with negative index. This allows us to introduce an integer-valued function $h^*(n)$ satisfying

$$\frac{\alpha_1 A_{21}}{1-\alpha_1} h^*(n) Q_1(h^*(n)) \sim Q_2(n). \tag{11}$$

Using properties of regularly varying functions (see, for instance, [14], Ch.1, Lemma 1.8), we get

$$h^*(n) = n^{\alpha_1/(\alpha_2(1-\alpha_1))} l_{h^*}(n), \tag{12}$$

where $l_{h^*}(n)$ is a function slowly varying at infinity. Moreover, $h^*(n) \ll n$ (see Lemma 1 page 9). In what follows equalities of the form $\mathbf{Z}(x,n) = y$ for $x, y \in (0,\infty)$ will be understood as $\mathbf{Z}([x], n) = [y]$ where $[w]$ stands for the integer part of w. The same agreement will be in force in other similar situations. For instance, $Q_i(x), x \in [0,\infty)$ will be treated as $Q_i([x])$.

Theorem 1. *Let Conditions* (2)-(4) *and* (8) *be satisfied and* (s_1, s_2) *be in* $(0,1)^2$.

(0) If $1 \le m \ll n$, *then*

$$\lim_{n\to\infty} \mathbf{E}\left[s_1^{Z_1(m,n)} s_2^{Z_2(m,n)} | \mathbf{Z}(n) \neq \mathbf{0} \right] = s_2; \tag{13}$$

(1) For any $a \in (0,1)$

$$\lim_{n\to\infty} \mathbf{E}\left[s_1^{Z_1(an,n)} s_2^{Z_2(an,n)} | \mathbf{Z}(n) \neq \mathbf{0} \right] = 1 - \left(a + (1-a)(1-s_2)^{-\alpha_2} \right)^{-\frac{1}{\alpha_2}}; \tag{14}$$

(2) For any function h *satisfying* $h^*(n) \ll h(n) \ll n$ *and* $m = n - h(n)$, *and* $\lambda_2 \ge 0$

$$\lim_{n\to\infty} \mathbf{E}\left[s_1^{Z_1(m,n)} \exp\left\{ -\lambda_2 \frac{Q_2(n)}{Q_{21}(h(n))} Z_2(m,n) \right\} | \mathbf{Z}(n) \neq \mathbf{0} \right] = 1 - \left(1 + \lambda_2^{-\alpha_2}\right)^{-1/\alpha_2};$$

(3) For any $t > 0$, $\lambda_2 \ge 0$ *and* $m = n - th^*(n)$

$$\lim_{n\to\infty} \mathbf{E}\left[s_1^{Z_1(m,n)} \exp\left\{ -\lambda_2 \frac{Q_2(n)}{Q_{21}(xh^*(n))} Z_2(m,n) \right\} | \mathbf{Z}(n) \neq \mathbf{0} \right]$$
$$= 1 - \left(1 + \left(t^{1-1/\alpha_1} (1-s_1)^{1-\alpha_1} + \lambda_2 \right)^{-\alpha_2} \right)^{-1/\alpha_2};$$

(4) For any function h satisfying $1 \ll h(n) \ll h^*(n)$, $m = n - h(n)$, and $\lambda_1, \lambda_2 \geq 0$

$$\lim_{n\to\infty} \mathbf{E}\left[\exp\left\{-\lambda_1 \frac{Q_1(h^*(n))}{Q_1(h(n))} Z_1(m,n) - \lambda_2 \frac{Q_2(n)}{Q_{21}(h(n))} Z_2(m,n)\right\} \Big| \mathbf{Z}(n) \neq \mathbf{0}\right]$$
$$= 1 - \left(1 + \left(\lambda_1^{1-\alpha_1} + \lambda_2\right)^{-\alpha_2}\right)^{-1/\alpha_2}.$$

We see that under the conditions of Theorem 1 the reduced process essentially consists of one particle of the second type at the initial stage, it consists of many second type particles but still no particles of the first type at the intermediate stage, and particles of the first type appear in the process only at time $m = n - th^*(n)$, $t > 0$. As a consequence, the coexistence of both types in the reduced process is possible only within a relatively small (in comparison with n) time interval of order $h^*(n)$, located at the end of the time frame under consideration, $[0, n]$. Thus, the evolution of this reduced process is mainly supported by particles of the initial type.

Let us now deal with Condition (9). To formulate the desired result we need to introduce a function $\phi = \phi(\lambda_1, \lambda_2), \lambda_1, \lambda_2 \geq 0$, which solves the partial differential equation

$$(1+\alpha_2)\lambda_1 \frac{\partial \phi}{\partial \lambda_1} + \lambda_2 \frac{\partial \phi}{\partial \lambda_2} = \phi - \phi^{1+\alpha_2} + \alpha_2 A_{21} \lambda_1 \tag{15}$$

with initial conditions

$$\phi(0,0) = 0, \quad \frac{\partial \phi(\lambda_1, \lambda_2)}{\partial \lambda_1}|_{\lambda_1=\lambda_2=0} = A_{21}, \quad \frac{\partial \phi(\lambda_1, \lambda_2)}{\partial \lambda_2}|_{\lambda_1=\lambda_2=0} = 1.$$

The problem of existence and uniqueness of a solution for Equation (15) has been solved in [24] and [11]. Note that if $\alpha_1 = \alpha_2 = 1$, the solution of (15) has an explicit form:

$$\phi(\lambda_1, \lambda_2) = \sqrt{A_{21}\lambda_1} \frac{\lambda_2 + \sqrt{A_{21}\lambda_1} \tanh \sqrt{A_{21}\lambda_1}}{\lambda_2 \tanh \sqrt{A_{21}\lambda_1} + \sqrt{A_{21}\lambda_1}}.$$

Let $g^*(n)$ be an integer-valued function satisfying

$$Q_2(g^*(n)) \sim Q_{21}(n). \tag{16}$$

Using properties of regularly varying functions and the asymptotic representation for $Q_{21}(n)$ given later on in (34) one may check that

$$g^*(n) = n^{\alpha_2/(\alpha_1(1+\alpha_2))} l_{g^*}(n), \tag{17}$$

where $l_{g^*}(n)$ is a function slowly varying at infinity, and that $g^*(n) \ll n$.

Theorem 2. *Let Conditions (2)-(4) and (9) be satisfied and (s_1, s_2) be in $(0,1)^2$.*

(0) If $1 \leq m \ll g^*(n)$, then

$$\lim_{n\to\infty} \mathbf{E}\left[s_1^{Z_1(m,n)} s_2^{Z_2(m,n)} | \mathbf{Z}(n) \neq \mathbf{0}\right] = s_2;$$

(1) For any $t > 0$

$$\lim_{n\to\infty} \mathbf{E}\left[s_1^{Z_1(tg^*(n),n)} s_2^{Z_2(tg^*(n),n)} | \mathbf{Z}(n) \neq \mathbf{0}\right] = 1 - t^{-1/\alpha_2} \phi\Big((1-s_1)\frac{t^{1/\alpha_2+1}}{\alpha_2 A_{21}}, (1-s_2)t^{1/\alpha_2}\Big);$$

(2) If $g^*(n) \ll m \ll n$, then

$$\lim_{n\to\infty} \mathbf{E}\left[s_1^{Z_1(m,n)} s_2^{Z_2(m,n)} | \mathbf{Z}(n) \neq \mathbf{0}\right] = 1 - (1-s_1)^{1/(1+\alpha_2)};$$

(3) For any $a \in (0,1)$

$$\lim_{n\to\infty} \mathbf{E}\left[s_1^{Z_1(an,n)} s_2^{Z_2(an,n)} | \mathbf{Z}(n) \neq \mathbf{0}\right] = 1 - \left(a + (1-a)(1-s_1)^{-\alpha_1}\right)^{-1/(\alpha_1(1+\alpha_2))}; \tag{18}$$

(4) If $m = n - h(n)$ for some $1 \ll h(n) \ll n$, then for $\lambda_1 \geq 0$

$$\lim_{n\to\infty} \mathbf{E}\left[\exp\left\{-\lambda_1 \frac{Q_1(n)}{Q_1(h(n))} Z_1(m,n)\right\} s_2^{Z_2(m,n)} | \mathbf{Z}(n) \neq \mathbf{0}\right] = 1 - \left(1+\lambda_1^{-\alpha_1}\right)^{-1/(\alpha_1(1+\alpha_2))}.$$

Once again, coexistence of both types in the reduced process is possible only within a relatively small time interval (here of order $g^*(n)$). But unlike the case of Theorem 1 this interval is located at the beginning of the population evolution and the reduced process is essentially composed of type 1 individuals.

The next theorem deals with Condition (10). Let $b \in (1,\infty)$ be the unique positive solution of

$$x^{1+\alpha_2} - x = \sigma\alpha_2 A_{21}, \tag{19}$$

and $\psi = \psi(\lambda_1,\lambda_2)$, $\lambda_1,\lambda_2 \geq 0$, the unique solution of the quasi-linear partial differential equation

$$(1+\alpha_2)\lambda_1 \frac{\partial\psi}{\partial\lambda_1} + \lambda_2 \frac{\partial\psi}{\partial\lambda_2} = \psi - b^{\alpha_2}\psi^{1+\alpha_2} + \alpha_2 A_{21}\lambda_1\left(1+\left(\frac{b}{\sigma}\lambda_1\right)^{\alpha_1}\right)^{-\frac{1}{\alpha_1}} \tag{20}$$

with initial conditions

$$\psi(0,0) = 0, \quad \frac{\partial\psi(\lambda_1,\lambda_2)}{\partial\lambda_1}|_{\lambda_1=\lambda_2=0} = A_{21}, \quad \frac{\partial\psi(\lambda_1,\lambda_2)}{\partial\lambda_2}|_{\lambda_1=\lambda_2=0} = 1. \tag{21}$$

Then we have the following asymptotics for the intermediate case (10):

Theorem 3. *Let Conditions (2)-(4) and (10) be satisfied, and (s_1,s_2) be in $(0,1)^2$.*

(0) If $1 \leq m \ll n$, then

$$\lim_{n\to\infty} \mathbf{E}\left[s_1^{Z_1(m,n)} s_2^{Z_2(m,n)} | \mathbf{Z}(n) \neq \mathbf{0}\right] = s_2; \tag{22}$$

(1) For any $a \in (0,1)$,

$$\lim_{n\to\infty} \mathbf{E}\left[s_1^{Z_1(an,n)} s_2^{Z_2(an,n)} \middle| \mathbf{Z}(n) \neq \mathbf{0}\right] = 1 - a^{-1/\alpha_2}\psi\Big((1-s_1)\frac{\sigma}{b}\Big(\frac{a}{1-a}\Big)^{1/\alpha_1}, (1-s_2)\Big(\frac{a}{1-a}\Big)^{1/\alpha_2}\Big);$$

(2) If $m = n - h(n)$, where $1 \ll h(n) \ll n$, then for $\lambda_1,\lambda_2 \geq 0$

$$\lim_{n\to\infty} \mathbf{E}\left[\exp\left\{-\frac{\lambda_1 Q_{21}(n)}{nQ_1(h(n))} Z_1(m,n) - \frac{\lambda_2 Q_2(n)}{Q_2(h(n))} Z_2(m,n)\right\} \middle| \mathbf{Z}(n) \neq \mathbf{0}\right] = 1 - \psi(\lambda_1,\lambda_2).$$

In this last case, both types of particles coexist during an interval of order n.

As we will see in Section 3, Theorems 1-3 will be elements of the proofs of more general statements dealing with conditional functional limit theorems for reduced branching processes.

3. Global description of the limit processes

In this section, we study in detail the dynamics of the reduced process, in particular, the transitions between monotype and two-type phases. Let $x^{[k]} = x(x-1)\cdots(x-k+1)$. In Definitions 1-2, we specify five auxiliary continuous time Markov processes needed to describe the transitions.

Definition 1. *We denote by $\{\nu = (\nu_1,\nu_2), (g_1(s_1),\mu_1), (g_{21}(s_1,s_2),\mu_2)\}$ a homogeneous Markov branching process initiated at time $t = 0$ by ν_1 particles of type 1 and ν_2 particles of type 2. At rate μ_1 (resp. μ_2) a particle of type 1 (resp. 2) dies and produces children in accordance with the probability generating function g_1 (resp. g_{21}). The newborn particles behave as their parents and independently of each other and of the past. Notice that when type 1 (resp. 2) particles do not evolve (no death, no offspring), $g_1(s_1) = s_1$ (resp. $g_{21}(s_1,s_2) = s_2$) and we can take*

any μ_1 (resp. μ_2). We will write μ_i, $i\in\{1,2\}$ to make it clear in the definition. With this representation in hands we can now introduce the following four branching processes:

(1) $\{X(t),t\geq 0\}=\{\mathbf{e}_2,(s_1,\mu_1),(g_2^{(X)}(s_1,s_2),1)\}$, *where*

$$g_2^{(X)}(s_1,s_2)=\frac{1}{\alpha_2}((1-s_2)^{1+\alpha_2}-1+(1+\alpha_2)s_2)=\frac{1}{\alpha_2}\sum_{k=2}^{\infty}\frac{(1+\alpha_2)^{[k]}}{k!}(-1)^k s_2^k. \tag{23}$$

(2) $\{\mathbf{Y}(t)=(Y_1(t),Y_2(t)),t\geq 0\}=\{\mathbf{e}_2,(s_1,\mu_1),(g_{21}^{(Y)}(s_1,s_2),1+1/\alpha_2)\}$, *where*

$$\begin{aligned} g_{21}^{(Y)}(s_1,s_2) &= \frac{1}{1+\alpha_2}\Big((1-s_2)^{1+\alpha_2}-1+(1+\alpha_2)s_2+s_1\Big)\\ &= \frac{1}{1+\alpha_2}\sum_{k=2}^{\infty}\frac{(1+\alpha_2)^{[k]}}{k!}(-1)^k s_2^k+\frac{1}{1+\alpha_2}s_1. \end{aligned}\tag{24}$$

(3) $\{V(t),t\geq 0\}=\{\mathbf{e}_1,(g_1^{(V)}(s_1),1),(s_2,\mu_2)\}$, *where*

$$g_1^{(V)}(s_1)=\frac{1}{\gamma_1-1}\Big[(1-s_1)^{\gamma_1}+\gamma_1 s_1-1\Big]=\frac{1}{\gamma_1-1}\sum_{k=2}^{\infty}\frac{\gamma_1^{[k]}}{k!}(-1)^k s_1^k, \tag{25}$$

and $\gamma_1=1+\alpha_1(1+\alpha_2)$.

(4) $\{\mathbf{W}(t)=(W_1(t),W_2(t)),t\geq 0\}=\{\mathbf{e}_2,(g_1^{(W)}(s_1),1),(g_{21}^{(W)}(s_1,s_2),\kappa)\}$, *where*

$$\kappa:\frac{(1+\alpha_2)b^{\alpha_2}-1}{\alpha_2}>1, \tag{26}$$

$$g_1^{(W)}(s_1)=\frac{1}{\alpha_1}\Big((1-s_1)^{1+\alpha_1}-1+(1+\alpha_1)s_1\Big)=\frac{1}{\alpha_1}\sum_{k=2}^{\infty}\frac{(1+\alpha_1)^{[k]}}{k!}(-1)^k s_1^k, \tag{27}$$

and

$$\begin{aligned} g_{21}^{(W)}(s_1,s_2) &= \frac{1}{\alpha_2\kappa}\Big(\frac{\sigma\alpha_2 A_{21}}{b}s_1+b^{\alpha_2}\Big((1-s_2)^{1+\alpha_2}-1+(1+\alpha_2)s_2\Big)\Big)\\ &= \frac{\sigma A_{21}}{b\kappa}s_1+\frac{b^{\alpha_2}}{\alpha_2\kappa}\sum_{k=2}^{\infty}\frac{(1+\alpha_2)^{[k]}}{k!}(-1)^k s_2^k. \end{aligned}\tag{28}$$

Note that $X(\cdot)$ and $V(\cdot)$ are monotype processes, while $\mathbf{Y}(\cdot)$ and $\mathbf{W}(\cdot)$ are two-type processes. Moreover, particles of type 1 of the process $\mathbf{Y}(\cdot)$ are sterile and immortal.

Definition 2. *We denote by $\{\mathbf{G}(t)=(G_1(t),G_2(t)),t\geq 0\}$ a two-dimensional Markov process with values in $\mathbf{N}_0\times\mathbf{R}_+=\{0,1,2,...\}\times(0,\infty)$. The initial state of $\mathbf{G}(\cdot)$ is random:*

$$\mathbf{G}(0)=(G_1(0),G_2(0))=(0,\theta_2)$$

where the distribution of θ_2 is given by

$$\mathbf{E}\left[e^{-\lambda\theta_2}\right]=1-\left(1+\lambda^{-\alpha_2}\right)^{-1/\alpha_2},\quad \lambda>0. \tag{29}$$

The distribution of $\mathbf{G}(t)$ at time $t>0$ is specified by

$$\mathbf{E}\left[s_1^{G_1(t)}e^{-\lambda_2 G_2(t)}\right]=1-\left(1+\left(t^{1/\alpha_1-1}(1-s_1)^{1-\alpha_1}+\lambda_2\right)^{-\alpha_2}\right)^{-1/\alpha_2},0\leq s_1\leq 1,\lambda_2>0,$$

and the transition probabilities for $0 \le t_0 < t_1 < \infty$ *and* $(n_1, y) \in \mathbf{N}_0 \times \mathbf{R}_+$ *are given by*

$$\mathbf{E}\left[s_1^{G_1(t_1)} e^{-\lambda_2 G_2(t_1)} | \, G_1(t_0) = n_1, G_2(t_0) = y\right] = \left(1 - \left[1 - \frac{t_1}{t_0} + \frac{t_1}{t_0}(1-s_1)^{-\alpha_1}\right]^{-1/\alpha_1}\right)^{n_1}$$
$$\times \exp\left\{-\left(t_1^{1-1/\alpha_1}(1-s_1)^{1-\alpha_1}\left(1 - \left(1 + \left(\frac{t_0}{t_1} - 1\right)(1-s_1)^{\alpha_1}\right)^{1-1/\alpha_1}\right) + \lambda_2\right) y\right\}.$$

In the sequel the symbol $\mathcal{L}_\mu$ will denote the law with initial condition μ which may be a measure or a random variable, and we will write $\mathcal{L}^{(n)}(\cdot)$ for $\mathcal{L}(\cdot|\mathbf{Z}(n) \neq \mathbf{0})$. Moreover, we say that a process $\{\mathbf{\Theta}(t), t \ge 0\}$ has a.s. constant paths with (random) value $\mathbf{\Theta}$ if

$$\mathbf{P}(\mathbf{\Theta}(t) = \mathbf{\Theta} \text{ for all } t \in (0, \infty)) = 1.$$

We have the following global description of the reduced process under Condition (8) and the assumption $\mathbf{Z}(0) = \mathbf{e}_2$, where the symbol $\Longrightarrow$ means convergence in the respective space of càdlàg functions endowed with Skorokhod topology.

Theorem 4. *Let Conditions* (2)-(4) *and* (8) *be satisfied.*

(1) *If* $m = \left[(1 - e^{-t})n\right]$, $t \ge 0$, *then*
$$\mathcal{L}^{(n)}\{\mathbf{Z}(m,n), 0 \le t < \infty\} \Longrightarrow \mathcal{L}_{(0,1)}\{(0, X(t)), 0 \le t < \infty\};$$

(2) *If* $m = n - th(n)$, $t > 0$, *for* $h^*(n) \ll h(n) \ll n$, *then*
$$\mathcal{L}^{(n)}\left\{\left(Z_1(m,n), \frac{Q_2(n)}{Q_{21}(h(n))} Z_2(m,n)\right), 0 < t < \infty\right\} \Longrightarrow \mathcal{L}\{(0, \theta_2(t)), 0 < t < \infty\},$$
where the limiting process has a.s. constant paths with value θ_2 *specified by (29);*

(3) *If* $m = n - th^*(n)$, $t > 0$, *then*
$$\mathcal{L}^{(n)}\left\{\left(Z_1(m,n), \frac{Q_2(n)}{Q_{21}(th^*(n))} Z_2(m,n)\right), 0 < t < \infty\right\} \Longrightarrow \mathcal{L}\{\mathbf{G}(t), 0 < t < \infty\};$$

(4) *For any* $\lambda_1, \lambda_2 > 0$, *any function* $h(n)$ *satisfying* $1 \ll h(n) \ll h^*(n)$ *and* $m = n - th(n)$
$$\mathcal{L}^{(n)}\left\{\left(\frac{Q_1(h^*(n))}{Q_1(th(n))} Z_1(m,n), \frac{Q_2(n)}{Q_{21}(th(n))} Z_2(m,n)\right), 0 < t < \infty\right\} \Longrightarrow \mathcal{L}\{\mathbf{\Theta}(t), 0 < t < \infty\},$$
where the limiting process has a.s. constant paths with value (θ_1, θ_2) *specified by*
$$\mathbf{E}\left[e^{-\lambda_1\theta_1} e^{-\lambda_2\theta_2}\right] = 1 - \left(1 + \left(\lambda_1^{1-\alpha_1} + \lambda_2\right)^{-\alpha_2}\right)^{-1/\alpha_2}.$$

Let d_n and q_n, $n = 1, 2, ...,$ be positive functions such that $d_n q_n = 1$ and
$$\lim_{n\to\infty} d_n = \infty \text{ and } d_n n^{-\varepsilon} = 0 \text{ for any } \varepsilon > 0.$$

Then we have the following result under Condition (9):

Theorem 5. *Let Conditions (2)-(4) and (9) be satisfied. Then*

(1)
$$\mathcal{L}^{(n)}\{\mathbf{Z}(tg^*(n), n), 0 \le t < \infty\} \Longrightarrow \mathcal{L}_{(0,1)}\{\mathbf{Y}(t), 0 \le t < \infty\}; \tag{30}$$

(2) *If* $\gamma_2 = \alpha_2/(\alpha_1(1+\alpha_2))$, *then*
$$\mathcal{L}^{(n)}\{\mathbf{Z}(d_n g^*(n) n^y, n), 0 \le y < 1 - \gamma_2\} \Longrightarrow \mathcal{L}_{(\theta_1, 0)}\left\{(\theta_1(y), 0), 0 \le y < 1 - \gamma_2\right\},$$
where $\theta_1(\cdot)$ *has a.s. constant paths with value* θ_1 *specified by*
$$\mathbf{E}\left[s^{\theta_1}\right] = 1 - (1-s)^{1/(1+\alpha_2)}, \quad 0 \le s \le 1. \tag{31}$$

(3) *If* $m = \left[\left((1-e^{-t}) + q_n\right) n\right]$, *then*

$$\mathcal{L}^{(n)}\{\mathbf{Z}(m,n), 0 \le t < \infty\} \Longrightarrow \mathcal{L}_{(\theta_1,0)}\{(V_1(t),0), 0 \le t < \infty\},$$

where the law of θ_1 *has been specified in* (31).

(4) *If* $1 \ll h(n) \ll n$ *and* $m = n - th(n)$ *then*

$$\mathcal{L}^{(n)}\left\{\left(\frac{Q_1(n)}{Q_1(th(n))}Z_1(m,n), Z_2(m,n)\right), 0 < t < \infty\right\} \Longrightarrow \mathcal{L}\{(\rho(t),0), 0 < t < \infty\},$$

where the limiting process has a.s. constant paths whose value ρ *is specified by*

$$\mathbf{E}\left[e^{-\lambda\rho}\right] = 1 - (1+\lambda^{-\alpha_1})^{-1/(\alpha_1(1+\alpha_2))}.$$

For the case $\alpha_1 = \alpha_2 = 1$, where $2g_{21}^{(Y)}(s_1,s_2) = s_2^2 + s_1$, this result is established in [20].

Finally, we have the following result under Condition (10) and the assumption $\mathbf{Z}(0) = \mathbf{e}_2$:

Theorem 6. *Let Conditions (2)-(4) and (10) be satisfied.*

(1) *If* $m = \left[(1-e^{-t})n\right]$, *then*

$$\mathcal{L}^{(n)}\{\mathbf{Z}(m,n), 0 \le t < \infty\} \Longrightarrow \mathcal{L}_{(0,1)}\{\mathbf{W}(t), 0 \le t < \infty\};$$

(2) *If* $n - m = th(n)$, *where* $1 \ll h(n) \ll n$, *then*

$$\mathcal{L}^{(n)}\left\{\left(\frac{Q_{21}(n)}{nQ_1(th(n))}Z_1(m,n), \frac{Q_2(n)}{Q_2(th(n))}Z_2(m,n)\right), 0 < t < \infty\right\} \Longrightarrow \mathcal{L}\{\mathbf{\Upsilon}(t), 0 < t < \infty\},$$

where the limiting process has a.s. constant paths with value (υ_1,υ_2) *specified by*

$$\mathbf{E}\left[e^{-\lambda_1\upsilon_1}e^{-\lambda_2\upsilon_2}\right] = 1 - \psi\left(\lambda_1,\lambda_2\right).$$

We end this section by the description of the distribution of the MRCA. More precisely, noticing that

$$\{Z_1(m,n) + Z_2(m,n) = 1\} = \{\beta_n \ge m\},$$

where the definition of β_n has been given in (1), we deduce from Theorems 4-6 the following statements concerning the birth time β_n and the type $\mathcal{T}_n$ of the MRCA:

Theorem 7. *Let Conditions (2)-(4) be satisfied.*

1) If Condition (8) is fulfilled then for any $a \in (0,1)$

$$\lim_{n\to\infty} \mathbf{P}\left(\beta_n \le an, \mathcal{T}_n = 2 | \mathbf{Z}(n) \neq \mathbf{0}\right) = a;$$

2) If Condition (9) is fulfilled, then for any $t \in (0,\infty)$ *and* $a \in (0,1)$,

$$\lim_{n\to\infty} \mathbf{P}_2\left(\beta_n \le tg^*(n), \mathcal{T}_n = 2\middle|\mathbf{Z}(n) \neq \mathbf{0}\right) = \frac{\alpha_2}{1+\alpha_2}(1-e^{-(1+\alpha_2)t/\alpha_2}),$$

and

$$\lim_{n\to\infty} \mathbf{P}_2(g^*(n) \ll \beta_n \le an, \mathcal{T}_n = 1 | \mathbf{Z}(n) \neq \mathbf{0}) = \frac{a}{1+\alpha_2}.$$

3) If Condition (10) is fulfilled, then for any $a \in (0,1)$

$$\lim_{n\to\infty} \mathbf{P}_2\left(\beta_n \le an | \mathbf{Z}(n) \neq \mathbf{0}\right) = 1 - \frac{1}{1+\alpha_2}(1-a) - \frac{\alpha_2}{1+\alpha_2}(1-a)^{((1+\alpha_2)b^{\alpha_2}-1)/\alpha_2}.$$

Moreover, the type of the MRCA satisfies:

$$\lim_{n\to\infty} \mathbf{P}_2\left(\mathcal{T}_n = 2 | \mathbf{Z}(n) \neq \mathbf{0}\right) = \frac{\alpha_2 b^{\alpha_2}}{(1+\alpha_2)b^{\alpha_2} - 1}.$$

Remark 1. *As a by-product of the proof of Theorem 2 (see Equation* (52)*) we can obtain a simple expression for the death moment* $\delta_n(2)$ *of the last ancestor of type 2,*

$$\delta_n(2) := \min\{m \leq n : Z_2(m,n) = 0\}.$$

Namely, under the conditions of Theorem 2 for any $t \in (0,\infty)$

$$\lim_{n\to\infty} \mathbf{P}_2\left(\delta_n(2) \leq tg^*(n) | \mathbf{Z}(n) \neq \mathbf{0}\right) = 1 - (1+t)^{-1/\alpha_2}.$$

Theorem 7 states that the law of the MRCA under Conditions (9) and (10) essentially differs from its law under Condition (8), where it is almost surely of type 2 and with birth time uniformly distributed on the time interval $[0,n]$.

4. Auxiliary results

In this section we list some known results and prove a number of auxiliary lemmas. The first statement, which will be useful at several occasions, is a direct consequence of the representation of regularly varying functions (see for example [7], Ch.VIII, Section 9).

Lemma 1. *Let* $R(n)$ *be a function regularly varying at infinity with index* $-\alpha < 0$*. If* $n \geq m \to \infty$ *then* $R(m) \gg R(n)$ *if and only if* $m \ll n$*.*

We now reformulate Theorem 1 in [28], formula (11) in [11] and Theorem 1 in [24] as a single statement. It will be an important tool in describing the local behavior for the reduced processes.

Theorem 8. *Let Conditions (2)-(4) be satisfied.*

(1) If Condition (8) *holds, then*

$$Q_{21}(n) \sim Q_2(n), \tag{32}$$

and for $\lambda_1 > 0, \lambda_2 > 0$

$$\lim_{n\to\infty} \mathbf{E}\left[\exp\{-\lambda_1 Q_1(h^*(n)) Z_1(n) - \lambda_2 Q_2(n) Z_2(n)\} | \mathbf{Z}(n) \neq \mathbf{0}\right]$$
$$= 1 - \left(1 + (\lambda_1^{1-\alpha_1} + \lambda_2)^{-\alpha_2}\right)^{-1/\alpha_2}; \tag{33}$$

(2) If Condition (9) *holds, then*

$$Q_{21}(n) \sim Q_1^{1/(1+\alpha_2)}(n) L_3(n), \quad Q_2(n) = o(Q_{21}(n)), \tag{34}$$

where $L_3(n)$ *is a slowly varying function as* $n \to \infty$*, and*

$$Q_{21}^{1+\alpha_2}(n) L_2(Q_{21}(n)) \sim A_{21} Q_1(n), \tag{35}$$

where L_2 *has been introduced in* (3)*. In addition, for all* $\lambda_1 \geq 0, \lambda_2 \geq 0$

$$\lim_{n\to\infty} \mathbf{E}\Big[\exp\{-\lambda_1 Q_1(n) Z_1(n)\} | \mathbf{Z}(n) \neq \mathbf{0}\Big] = 1 - (1 + \lambda_1^{-\alpha_1})^{-1/(\alpha_1(1+\alpha_2))},$$

and

$$\lim_{n\to\infty} \frac{1}{Q_2(n)} \mathbf{E}_2\left[1 - \exp\left\{-\lambda_1 \frac{Q_2(n)}{n} Z_1(n) - \lambda_2 Q_2(n) Z_2(n)\right\}\right] = \phi(\lambda_1,\lambda_2). \tag{36}$$

(3) If Condition (10) *holds, then*

$$Q_{21}(n) \sim b Q_2(n), \tag{37}$$

where $b \in (1,\infty)$ *has been defined in* (19)*. In addition, for all* $\lambda_1 \geq 0, \lambda_2 \geq 0$

$$\lim_{n\to\infty} \frac{1}{Q_{21}(n)} \mathbf{E}_2\left[1 - \exp\left\{-\lambda_1 \frac{Q_{21}(n)}{n} Z_1(n) - \lambda_2 Q_{21}(n) Z_2(n)\right\}\right] = \psi(\lambda_1,\lambda_2). \tag{38}$$

To simplify notations we agree to write $\mathbf{s}^{\mathbf{k}} = s_1^{k_1} s_2^{k_2}$ for any vector $\mathbf{s} = (s_1, s_2)$ and integer-valued vector $\mathbf{k} = (k_1, k_2)$. Besides, for vectors $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ we set $\mathbf{x} \otimes \mathbf{y} = (x_1 y_1, x_2 y_2)$ and let $\mathbf{1} = (1, 1)$. Let us now present a way of expressing the probability generating function of the reduced process which will be repeatedly used in the proofs and will allow us to apply Theorem 7. To do this, we need to introduce some notation. Put, for $i \in \{1, 2\}$

$$\begin{array}{ll} F_i(n; s_i) = \mathbf{E}\left[s_i^{Z_i(n)} | \mathbf{Z}(0) = \mathbf{e}_i\right], & F_{21}(n; \mathbf{s}) = \mathbf{E}\left[\mathbf{s}^{\mathbf{Z}(n)} | \mathbf{Z}(0) = \mathbf{e}_2\right], \\ Q_i(n; s_i) = 1 - F_i(n; s_i), & Q_{21}(n; \mathbf{s}) = 1 - F_{21}(n; \mathbf{s}). \end{array}$$

By conditioning on $\mathbf{Z}(m)$, we can write for $m \leq n$

$$\mathbf{E}_2\left[\mathbf{s}^{\mathbf{Z}(m,n)}\right] = F_{21}(m; 1 - (1 - s_1)Q_1(n - m), 1 - (1 - s_2)Q_{21}(n - m)).$$

Hence, setting $\mathbf{Q}_{21}(k) = (Q_1(k), Q_{21}(k))$, $k = 0, 1, ...$, we get

$$1 - \mathbf{E}_2\left[\mathbf{s}^{\mathbf{Z}(m,n)} | \mathbf{Z}(n) \neq \mathbf{0}\right] = (Q_{21}(m; \mathbf{1} - (\mathbf{1} - \mathbf{s}) \otimes \mathbf{Q}_{21}(n - m)))/Q_{21}(n). \tag{39}$$

Thus, to prove Theorems 1-3 it is necessary to find the limit of the right-hand side of (39) under an appropriate choice of m, n and $\mathbf{s}$. If $n - m \to \infty$ it is equivalent to finding

$$\lim_{m,n\to\infty} Q_{21}(m; \mathbf{e}^{-(\mathbf{1}-\mathbf{s})\otimes\mathbf{Q}_{21}(n-m)})/Q_{21}(n). \tag{40}$$

The end of this section deals with several expressions similar to (39) or (40). These results will be needed at several occasions in the remaining sections.

Lemma 2. *If $m \ll n$, then, for $\lambda > 0$ and $i \in \{1, 2\}$*

$$Q_i(m; e^{-\lambda Q_i(n)}) \sim \lambda Q_i(n) \sim Q_i(m; 1 - \lambda Q_i(n)). \tag{41}$$

Proof. Let $h_\lambda(n)$ be the integer-valued function satisfying

$$Q_i(h_\lambda(n)) \leq 1 - e^{-\lambda Q_i(n)} \leq Q_i(h_\lambda(n) - 1).$$

We know by (5) that $Q_i(h_\lambda(n)) \sim Q_i(h_\lambda(n) - 1)$ and

$$1 - e^{-\lambda Q_i(n)} \sim \lambda Q_i(n) \sim Q_i(n\lambda^{-1/\alpha_i}), \tag{42}$$

implying $h_\lambda(n) \sim n\lambda^{-1/\alpha_i}$ as $n \to \infty$. Using the branching property we have

$$\begin{aligned} Q_i(m + h_\lambda(n)) = Q_i(m, F_i(h_\lambda(n), 0)) &\leq Q_i(m; e^{-\lambda Q_i(n)}) \\ &\leq Q_i(m, F_i(h_\lambda(n) - 1, 0)) = Q_i(m + h_\lambda(n) - 1). \end{aligned}$$

Since $m \ll h_\lambda(n)$, we get, again by (5) that, as $n \to \infty$

$$Q_i\left(m; e^{-\lambda Q_i(n)}\right) \sim Q_i(m + h_\lambda(n)) \sim Q_i(h_\lambda(n)) \sim Q_i\left(n\lambda^{-1/\alpha_i}\right) \sim \lambda Q_i(n)$$

proving the first equivalence in (41). The second equivalence follows from (42). ■

Lemma 3 deals with asymptotic results on the time scale $h^*(n)$ specified in (11).

Lemma 3. *Let Conditions (2)-(4) and (8) be satisfied and λ_1, λ_2, a be positive. Define*

$$s = 1 - \lambda_1 Q_1(h^*(n)). \tag{43}$$

Then we have the following two equivalences for large n,

$$\sum_{k=0}^{n-1} Q_1(k; s) \sim \frac{\lambda_1^{1-\alpha_1}}{A_{21}} Q_2(n), \qquad \sum_{k=0}^{ah^*(n)-1} Q_1(k; s) \sim \frac{\lambda_1^{1-\alpha_1}}{A_{21}} \left(1 - (1 + a\lambda_1^{\alpha_1})^{1-1/\alpha_1}\right) Q_2(n), \tag{44}$$

where we agree to understand $ah^*(n)$ *as* $[ah^*(n)]$. *Moreover,*

$$\lim_{n\to\infty} \frac{Q_{21}(ah^*(n);1-\lambda_1 Q_1(h^*(n)),1-\lambda_2 Q_2(n))}{Q_2(n)} = \lambda_1^{1-\alpha_1}\Big(1-(1+a\lambda_1^{\alpha_1})^{1-1/\alpha_1}\Big)+\lambda_2. \quad (45)$$

Proof. We do not prove the first assertion in (44), as it may be checked in a similar way as the second one. As in the proof of the previous lemma, if $h_\lambda(n)$ is an integer-valued function satisfying

$$Q_1(h_{\lambda_1}(n)) \le 1-s \le Q_1(h_{\lambda_1}(n)-1),$$

then

$$Q_1(k+h_{\lambda_1}(n)) \le Q_1(k;s) \le Q_1(k+h_{\lambda_1}(n)-1) \quad (46)$$

for every integer k. Equations (43) and (5) entail that, as $n\to\infty$

$$h_{\lambda_1}(n) \sim \lambda_1^{-\alpha_1} h^*(n). \quad (47)$$

By (46) we deduce the inequality

$$0 \le \sum_{k=0}^{ah^*(n)-1} Q_1(k;s_1) - \sum_{k=h_{\lambda_1}(n)}^{ah^*(n)+h_{\lambda_1}(n)-1} Q_1(k) \le Q_1(h_{\lambda_1}(n)-1).$$

Note that in view of (5), as $z\to\infty$

$$\sum_{k=z}^{\infty} Q_1(k) \sim \frac{\alpha_1}{1-\alpha_1} z Q_1(z) \sim \frac{1}{1-\alpha_1} z^{1-1/\alpha_1} l_1(z).$$

Hence we conclude that

$$\begin{aligned}
\sum_{k=h_{\lambda_1}(n)}^{ah^*(n)+h_{\lambda_1}(n)-1} Q_1(k) &= \sum_{k=h_{\lambda_1}(n)}^{\infty} Q_1(k) - \sum_{k=ah^*(n)+h_{\lambda_1}(n)}^{\infty} Q_1(k)\\
&\sim \frac{\alpha_1}{1-\alpha_1}\Big(h_{\lambda_1}(n)Q_1(h_{\lambda_1}(n)) - (ah^*(n)+h_{\lambda_1}(n))Q_1(ah^*(n)+h_{\lambda_1}(n))\Big)\\
&\sim \frac{\alpha_1}{1-\alpha_1}\lambda_1^{1-\alpha_1}\Big(1-(1+a\lambda_1^{\alpha_1})^{1-1/\alpha_1}\Big)h^*(n)Q_1(h^*(n))\\
&\sim \frac{1}{A_{21}}\lambda_1^{1-\alpha_1}\Big(1-(1+a\lambda_1^{\alpha_1})^{1-1/\alpha_1}\Big)Q_2(n),
\end{aligned}$$

where we have applied (47), (5) and (11).

We now prove (45). By definition, for $\mathbf{s}=(s_1,s_2)\in[0,1]^2$ and $k\in\mathbb{N}_0$,

$$Q_{21}(k+1;\mathbf{s}) = Q_{21}(k;\mathbf{s}) - Q_{21}^{1+\alpha_2}(k;\mathbf{s})L_2\left(Q_{21}(k;\mathbf{s}) + \left(A_{21}-\rho\left(k;\mathbf{s}\right)\right)Q_1(k;s_1\right),$$

where

$$\rho\left(k;\mathbf{s}\right) = \rho\left(F_1(k;s_1),F_{21}(k;\mathbf{s})\right) \to 0$$

as $k\to\infty$ uniformly in $\mathbf{s}\in[0,1]^2$. Thus,

$$Q_{21}(ah^*(n);\mathbf{s}) = Q_{21}(0;\mathbf{s}) - \sum_{k=0}^{ah^*(n)-1}\Big[Q_{21}^{1+\alpha_2}(k;\mathbf{s})L_2\left(Q_{21}(k;\mathbf{s})\right) - \left(A_{21}-\rho\left(k;\mathbf{s}\right)\right)Q_1(k;s_1)\Big].$$

Let, for sufficiently large n

$$s_1 = 1-\lambda_1 Q_1(h^*(n)) \quad \text{and} \quad s_2 = 1-\lambda_2 Q_2(n).$$

Using the inequality $1-c_1c_2 \le (1-c_1)+(1-c_2)$ for $c_1,c_2\in[0,1]$, we get

$$Q_{21}(k;\mathbf{s}) \le \mathbf{E}_2\left[Z_1(k)\right](\mathbf{1}-\mathbf{s}_1)+\mathbf{1}-\mathbf{s}_2 = A_{21}k\left(1-s_1\right)+1-s_2.$$

Adding (32) and (12), we obtain for $k \leq ah^*(n)$ and some C_1 independent of n,

$$Q_{21}(k;\mathbf{s}) \leq \lambda_1 A_{21} k Q_1(h^*(n)) + \lambda_2 Q_2(n) \leq C_1 Q_2(n).$$

This, in view of the monotonicity of $y^{1+\alpha_2} L_2(y)$ as $y \downarrow 0$ and $h^*(n) \ll n$ yields

$$\sum_{k=0}^{ah^*(n)-1} Q_{21}^{1+\alpha_2}(k;\mathbf{s}) L_2\left(Q_{21}(k;\mathbf{s})\right) \leq C_2 h^*(n) Q_2^{1+\alpha_2}(n) L_2(Q_2(n)) \sim C_2 \frac{h^*(n)Q_2(n)}{\alpha_2 n} = o(Q_2(n)),$$

where C_2 is finite and independent of n and we have applied (6) to get the equivalence. Finally, recalling that

$$\sum_{k=0}^{ah^*(n)-1} \left(A_{21} - \rho\left(k;\mathbf{s}\right)\right) Q_1(k;s_1) \sim \lambda_1^{1-\alpha_1}\Big(1 - (1 + a\lambda_1^{\alpha_1})^{1-1/\alpha_1}\Big) Q_2(n)$$

by (44) and that

$$Q_{21}(0;\mathbf{s}) = 1 - s_2 = \lambda_2 Q_2(n)$$

we get (45) and end the proof of the lemma. ■

We now focus on time scales $h(n)$ larger than $h^*(n)$, still under Condition (8).

Lemma 4. *Let Conditions (2)-(4) and (8) be satisfied and $h(n)$ be an integer-valued function such that $h^*(n) \ll h(n)$. Then, for any $s \in [0,1)$*

$$\lim_{n\to\infty} \sup_{m \leq n-h(n)} \mathbf{E}_2\left[1 - s^{Z_1(m,n)} | \mathbf{Z}(n) \neq \mathbf{0}\right] = \lim_{n\to\infty} \sup_{m\leq n-h(n)} \frac{\mathbf{E}_2\left[1 - s^{Z_1(m,n)}\right]}{Q_{21}(n)} = 0.$$

Proof. Since $Z_1(m,n)$ is monotone increasing in m given n, it is sufficient to consider $m = n - h(n)$. By conditioning on $Z_1(m)$ we get

$$\mathbf{E}_2\left[s^{Z_1(m,n)}\right] = \mathbf{E}_2\left[\left(F_1(n-m;0) + s\left(1 - F_1(n-m;0)\right)\right)^{Z_1(m)}\right] \geq \mathbf{E}_2\left[F_1^{Z_1(m)}(n-m;0)\right].$$

Let $\zeta(k), k = 1, 2, ...$ be the total amount of type 1 particles at moment k produced by all particles of type 2 existing in the branching process at moment $k-1$. Then,

$$\mathbf{E}_2\left[F_1^{Z_1(m)}(n-m;0)\right] = \mathbf{E}_2\left[\prod_{k=0}^{m-1} F_1^{\zeta(k)}(m-k; F_1(n-m;0))\right] = \mathbf{E}_2\left[\prod_{k=0}^{m-1} F_1^{\zeta(k)}(n-k;0)\right].$$

By iteration, we also deduce for every $k = 1, 2, ...$:

$$\mathbf{E}_2\left[\zeta(k)\right] = \mathbf{E}_2\left[\mathbf{E}_2\left[\zeta(k)|Z_2(k-1)\right]\right] = A_{21}\mathbf{E}_2\left[Z_2(k-1)\right] = A_{21}.$$

Using the estimates above we obtain

$$\begin{aligned} \mathbf{E}_2\left[1 - s^{Z_1(m,n)}\right] &\leq \mathbf{E}_2\left[1 - \prod_{k=0}^{m-1} F_1^{\zeta(k)}(n-k;0)\right] \leq \mathbf{E}_2\left[\sum_{k=0}^{m-1} \zeta(k) Q_1(n-k)\right] \\ &\leq A_{21} \sum_{k=n-m}^{\infty} Q_1(k) = O\left(h(n)Q_1(h(n))\right), \end{aligned}$$

where at the last stage we have used (5) and the fact that, as $n \to \infty$

$$Q_1(n) = o\left(n^{-1}Q_2(n)\right) = o\left(n^{-1-\varepsilon}\right)$$

for some $\varepsilon > 0$. Since $h^*(n) \ll h(n)$ as $n \to \infty$, we have by Lemma 1, (11) and (32) that

$$h(n)Q_1(h(n)) = o\left(h^*(n)Q_1(h^*(n))\right) = o\left(Q_{21}(n)\right),$$

proving the lemma. ■

We now state asymptotic results on the time scale $g^*(n)$ specified by (16).

Lemma 5. *Let Conditions (2)-(4) and (9) be satisfied and* (s_1,s_2) *be in* $[0,1]^2$*. Then*

(1)

$$Q_2(g^*(n)) \sim \alpha_2 A_{21} Q_1(n) g^*(n); \tag{48}$$

(2) If $m \ll g^*(n)$*, then*

$$\mathbf{E}_2\left[1-s_1^{Z_1(m,n)}\right] = o(Q_{21}(n)); \tag{49}$$

(3) If $g^*(n) \ll m \le n$*, then*

$$\mathbf{E}_2\left[1-s_2^{Z_2(m,n)}\right] = o(Q_{21}(n)). \tag{50}$$

Proof. First observe that according to (17), $g^*(n) \ll n$, and that (48) is a direct consequence of (6) and (35). To prove the remaining statements note that by the branching property

$$\mathbf{E}_2\left[1-s_1^{Z_1(m,n)}\right] \le \mathbf{E}_2\left[Z_1(m,n)\right] = \mathbf{E}_2\left[Z_1(m)\right] Q_1\left(n-m\right) = A_{21} m Q_1\left(n-m\right). \tag{51}$$

Recalling the condition $m \ll g^*(n)$, Lemma 1 and (48), we conclude that

$$mQ_1\left(n-m\right) \ll g^*(n) Q_1\left(n-g^*(n)\right) \sim g^*(n) Q_1\left(n\right) \sim \frac{Q_2(g^*(n))}{\alpha_2 A_{21}} \sim \frac{Q_{21}(n)}{\alpha_2 A_{21}}.$$

This proves (49). To justify (50) it is sufficient to observe that, for $g^*(n) \ll m$,

$$\mathbf{E}_2\left[1-s_2^{Z_2(m,n)}\right] \le Q_2(m) \ll Q_2(g^*(n)) \sim Q_{21}(n).$$

This ends the proof of the lemma. ■

To end this section we mention a classical result on branching processes. Recall Definition 1 and define $\{\mathbf{M}=(M_1(t),M_2(t)), t\ge 0\} = \{\nu, (g_1(s_1),\mu_1), (g_{21}(\mathbf{s}),\mu_2)\}$. Let

$$f_1(t;s_1) = \mathbf{E}\Big[s_1^{M_1(t)}|\mathbf{M}(0)=\mathbf{e}_1\Big], \quad f_{21}(t;\mathbf{s}) = \mathbf{E}\Big[\mathbf{s}^{\mathbf{M}(t)}|\mathbf{M}(0)=\mathbf{e}_2\Big].$$

Then we have the following statement (see, for instance, [12], Ch. IV, Section 3 or [1], Ch. V, Section 7).

Lemma 6. *The pair of functions* (f_1,f_{21}) *is the unique solution of*

$$\begin{cases} \frac{\partial f_{21}(t;\mathbf{s})}{\partial t} &= \mu_2\left(g_{21}\left(f_1\left(t;s_1\right), f_{21}\left(t;\mathbf{s}\right)\right) - f_{21}\left(t;\mathbf{s}\right)\right), \\ \frac{\partial f_1(t;s_1)}{\partial t} &= \mu_1(g_1(f_1(t;s_1)) - f_1(t;s_1)) \end{cases}$$

with initial conditions $f_1\left(0;s_1\right)=s_1$ *and* $f_{21}(0;\mathbf{s})=s_2$.

5. Proof of Theorem 1

Throughout this section we assume Conditions (2)-(4) and (8), and that (s_1,s_2) belongs to $(0,1)^2$.

Point (0): Let $m \ll n$. In view of (8) and Lemma 4

$$0 \le \mathbf{E}\left[s_2^{Z_2(m,n)}|\mathbf{Z}(n)\neq\mathbf{0}\right] - \mathbf{E}\left[\mathbf{s}^{\mathbf{Z}(m,n)}|\mathbf{Z}(n)\neq\mathbf{0}\right] \le \mathbf{E}\left[1-s_1^{Z_1(m,n)}|\mathbf{Z}(n)\neq\mathbf{0}\right] = o(1).$$

Thus,

$$\mathbf{E}\left[\mathbf{s}^{\mathbf{Z}(m,n)}|\mathbf{Z}(n)\neq\mathbf{0}\right] = 1-\mathbf{E}\left[1-s_2^{Z_2(m,n)}\right]/Q_{21}(n)+o(1).$$

Further, using the branching property and Lemma 2 we get

$$\mathbf{E}\left[1-s_2^{Z_2(m,n)}\right]=Q_2\left(m;1-(1-s_2)\,Q_2(n-m)\right)\sim(1-s_2)\,Q_2(n-m)\sim(1-s_2)\,Q_2(n).$$

This, on account of (32) implies (13).

Point (1): Let a be in $(0,1)$. As in the previous case, by (8) and Lemma 4, as $n\to\infty$,

$$\mathbf{E}\left[\mathbf{s}^{\mathbf{Z}(an,n)}|\mathbf{Z}(n)\neq\mathbf{0}\right]=\mathbf{E}\left[s_2^{Z_2(an,n)}|\mathbf{Z}(n)\neq\mathbf{0}\right]+o(1).$$

Moreover, in view of (7) and (32), as $n\to\infty$

$$\begin{aligned}1-\mathbf{E}\left[s_2^{Z_2(an,n)}|\mathbf{Z}(n)\neq\mathbf{0}\right]&=\frac{Q_2\left(an;1-(1-s_2)Q_{21}((1-a)n)\right)}{Q_{21}(n)}\\&\sim\frac{Q_2\left(an,e^{-(1-s_2)(a/(1-a))^{1/\alpha_2}Q_2(an)}\right)}{a^{1/\alpha_2}Q_2(an)}\\&\sim a^{-1/\alpha_2}\left(1+\left((1-s_2)\left(\frac{a}{1-a}\right)^{1/\alpha_2}\right)^{-\alpha_2}\right)^{-1/\alpha_2}\\&=\left(a+(1-a)(1-s_2)^{-\alpha_2}\right)^{-1/\alpha_2}.\end{aligned}$$

This completes the proof of point (1).

Point (2): Let $n-m=h(n)$, where $h(n)$ is an integer-valued function such that $h^*(n)\ll h(n)\ll n$. Similarly to the previous point we have by (8) and Lemma 4

$$0\leq\mathbf{E}\left[s_2^{Z_2(m,n)}|\mathbf{Z}(n)\neq\mathbf{0}\right]-\mathbf{E}\left[\mathbf{s}^{\mathbf{Z}(m,n)}|\mathbf{Z}(n)\neq\mathbf{0}\right]=o(1)$$

as $n\to\infty$. Now following the line of proving point (1) and letting

$$1-s_2=1-\exp\left\{-\lambda_2\frac{Q_2(n)}{Q_{21}(h(n))}\right\}\sim\lambda_2\frac{Q_2(n)}{Q_{21}(h(n))}$$

we get, using again (7),

$$\begin{aligned}\lim_{n\to\infty}\mathbf{E}\left[1-\mathbf{s}^{\mathbf{Z}(n-h(n),n)}\Big|\mathbf{Z}(n)\neq\mathbf{0}\right]&=\lim_{n\to\infty}\frac{Q_2(n-h(n);1-(1-s_2)Q_{21}(h(n)))}{Q_{21}(n)}\\&=\lim_{n\to\infty}\frac{Q_2(n-h(n);1-\lambda_2Q_2(n))}{Q_2(n)}=\left(1+\lambda_2^{-\alpha_2}\right)^{-1/\alpha_2}.\end{aligned}$$

Point (3): Let t be positive and $m=n-th^*(n)$. We need to find the limit of the right hand side of (39) with

$$1-s_2=\exp\left\{-\lambda_2\frac{Q_2(n)}{Q_{21}(th^*(n))}\right\}\sim\lambda_2\frac{Q_2(n)}{Q_{21}(th^*(n))}.$$

From (33) we know that

$$\lim_{m\to\infty}\frac{Q_{21}(m;e^{-\lambda_1Q_1(h^*(m))};e^{-\lambda_2Q_2(m)})}{Q_{21}(m)}=\left(1+\left(\lambda_1^{1-\alpha_1}+\lambda_2\right)^{-\alpha_2}\right)^{-1/\alpha_2}.$$

Hence, by taking a fixed $s_1\in[0,1)$ and adding (5) we deduce for $m=n-th^*(n)$ that

$$\begin{aligned}&\lim_{n\to\infty}\frac{Q_{21}(n-th^*(n);\mathbf{1}-(\mathbf{1}-\mathbf{s})\otimes\mathbf{Q}_{21}(th^*(n)))}{Q_{21}(n)}\\&\quad=\lim_{n\to\infty}\frac{Q_{21}(n-th^*(n);1-t^{-1/\alpha_1}(1-s_1)Q_1(h^*(n)),1-\lambda_2Q_2(n))}{Q_{21}(n)}\\&\qquad=(1+(t^{1-1/\alpha_1}(1-s_1)^{1-\alpha_1}+\lambda_2)^{-\alpha_2})^{-1/\alpha_2},\end{aligned}$$

which is the statement of point (3).

Point (4): Take $n-m=h(n)$ with $1\ll h(n)\ll h^*(n)$,

$$s_1=\exp\left\{-\lambda_1\frac{Q_1(h^*(n))}{Q_1(h(n))}\right\}\quad\text{and}\quad s_2=\exp\left\{-\lambda_2\frac{Q_2(n)}{Q_{21}(h(n))}\right\}.$$

Using (33) we get

$$\begin{aligned}\lim_{n\to\infty}\frac{Q_{21}(n-h(n);\mathbf{1}-(\mathbf{1}-\mathbf{s})\otimes\mathbf{Q}_{21}(h(n)))}{Q_{21}(n)} &= \lim_{n\to\infty}\frac{Q_{21}(n-h(n);e^{-\lambda_1Q_1(h^*(n))},e^{-\lambda_2Q_2(n)})}{Q_{21}(n-h(n))}\\ &= \left(1+(\lambda_1^{1-\alpha_1}+\lambda_2)^{-\alpha_2}\right)^{-1/\alpha_2}.\end{aligned}$$

This ends the proof of Theorem 1.

6. Proof of Theorem 2

Throughout this section we assume that Conditions (2)-(4) and (9) are in force and that (s_1,s_2) belongs to $(0,1)^2$.

Point (0): Let $m\ll g^*(n)$ with $g^*(n)$ specified by (16). Applying (49) we get, as $n\to\infty$

$$0\le\mathbf{E}\left[1-\mathbf{s}^{\mathbf{Z}(m,n)}\right]-\mathbf{E}\left[1-s_2^{Z_2(m,n)}\right]\le\mathbf{E}\left[1-s_1^{Z_1(m,n)}\right]=o\left(Q_{21}(n)\right).$$

Further, recalling (16) and Lemma 2 we see that

$$\begin{aligned}\mathbf{E}\left[1-s_2^{Z_2(m,n)}\right] &= Q_2\left(m;1-(1-s_2)Q_{21}(n-m)\right)\\ &\sim Q_2\left(m;1-(1-s_2)Q_2(g^*(n))\right)\sim(1-s_2)Q_2(g^*(n))\sim(1-s_2)Q_{21}(n).\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\mathbf{E}[1-s_2^{Z_2(m,n)}|\mathbf{Z}(n)\ne\mathbf{0}]=\lim_{n\to\infty}\frac{\mathbf{E}[1-\mathbf{s}^{\mathbf{Z}(m,n)}]}{Q_{21}(n)}=\lim_{n\to\infty}\frac{\mathbf{E}[1-s_2^{Z_2(m,n)}]}{Q_{21}(n)}=1-s_2,$$

which proves point (0).

Point (1): In view of (5) and (16), for any $t>0$

$$Q_{21}(n-tg^*(n))\sim Q_{21}(n)\sim Q_2(g^*(n))\sim t^{1/\alpha_2}Q_2(tg^*(n))$$

and by (48) and (5)

$$Q_1(n-tg^*(n))\sim Q_1(n)\sim\frac{Q_2(g^*(n))}{\alpha_2A_{21}g^*(n)}\sim\frac{t^{1/\alpha_2+1}}{\alpha_2A_{21}}\frac{Q_2(tg^*(n))}{tg^*(n)}.$$

Then, by using (39), (16) and (36) with n replaced by $tg^*(n)$ we obtain

$$\begin{aligned}\mathbf{E}_2[1-\mathbf{s}^{\mathbf{Z}(tg^*(n),n)}|\mathbf{Z}(n)\ne\mathbf{0}] &= \frac{Q_{21}(tg^*(n),1-(\mathbf{1}-\mathbf{s})\otimes\mathbf{Q}_{21}(n-tg^*(n)))}{Q_{21}(n)}\\ &\sim\frac{\mathbf{E}_2\left[1-\exp\left\{-(1-s_1)\frac{t^{1/\alpha_2+1}}{\alpha_2A_{21}}\frac{Q_2(tg^*(n))}{tg^*(n)}Z_1(tg^*(n))-(1-s_2)t^{1/\alpha_2}Q_2(tg^*(n))Z_2(tg^*(n))\right\}\right]}{t^{1/\alpha_2}Q_2(tg^*(n))}\\ &\sim t^{-1/\alpha_2}\phi\left((1-s_1)\frac{t^{1/\alpha_2+1}}{\alpha_2A_{21}},(1-s_2)t^{1/\alpha_2}\right)\end{aligned}$$

justifying the statement of point (1).

Point (3): If $g^*(n)\ll m\le n$ then, according to (50)

$$\left|\mathbf{E}_2\left[\left(1^{Z_2(m,n)}-0^{Z_2(m,n)}\right)s_1^{Z_1(m,n)}\right]\right|\le\mathbf{E}_2\left[1-0^{Z_2(m,n)}\right]=o\left(Q_{21}(n)\right).$$

Hence we obtain the same limit for (18) independently of the choice of s_2. We choose s_2 satisfying

$$(1-s_2)Q_{21}((1-a)n)\sim Q_{21}\left(\frac{(1-a)n}{(1-s_1)^{\alpha_1}}\right)\sim(1-s_1)^{1/(1+\alpha_2)}(1-a)^{-1/(\alpha_1(1+\alpha_2))}Q_{21}(n),$$

where we have applied (34). Then, using the branching property, (5) and (34), we get

$$\frac{Q_{21}(an;\mathbf{1}-(\mathbf{1}-\mathbf{s})\otimes\mathbf{Q}_{21}((1-a)n))}{Q_{21}(n)}\sim\frac{Q_{21}(an;\mathbf{1}-\mathbf{Q}_{21}((1-a)n/(1-s_1)^{\alpha_1}))}{Q_{21}(n)}$$
$$=\frac{Q_{21}(an+(1-a)n/(1-s_1)^{\alpha_1}))}{Q_{21}(n)}\sim\Big(a+\frac{(1-a)}{(1-s_1)^{\alpha_1}}\Big)^{-1/(\alpha_1(1+\alpha_2))}.$$

Adding (39), this proves point (3).

Point (2): We know from point (1) that for every $t>0$,

$$\lim_{n\to\infty}\mathbf{P}\left(Z_2(tg^*(n),n)=0|\mathbf{Z}(n)\neq\mathbf{0}\right)=1-t^{-1/\alpha_2}\phi(0,t^{1/\alpha_2}).$$

Setting $q(x)=\phi(0,x)/x$, $x>0$, we deduce by applying (15) that $q(x)$ satisfies

$$q'(x)=-x^{\alpha_2-1}q^{1+\alpha_2}(x),$$

with initial condition $q(0)=\lim_{x\downarrow 0}q(x)=1$. This equation has an explicit solution and we obtain

$$\lim_{n\to\infty}\mathbf{P}_2\left(Z_2(tg^*(n),n)=0|\mathbf{Z}(n)\neq\mathbf{0}\right)=1-q(t^{1/\alpha_2})=1-(1+t)^{-1/\alpha_2}. \tag{52}$$

In particular,

$$\lim_{t\to\infty}\lim_{n\to\infty}\mathbf{P}_2\left(Z_2(tg^*(n),n)=0|\mathbf{Z}(n)\neq\mathbf{0}\right)=1, \tag{53}$$

and for $m\gg g^*(n)$,

$$\lim_{m,n\to\infty}\mathbf{P}_2\left(Z_2(m,n)=0|\mathbf{Z}(n)\neq\mathbf{0}\right)=1. \tag{54}$$

We now take $r(x)=\phi(x,0)/x^{1/(1+\alpha_2)}, x>0$. Then from point (1) we have

$$\lim_{n\to\infty}\mathbf{E}_2\left[s_1^{Z_1((\alpha_2A_{21}x/(1-s_1))^{\alpha_2/(1+\alpha_2)}g^*(n),n)}|\mathbf{Z}(n)\neq\mathbf{0}\right]=1-\Big(\frac{1-s_1}{\alpha_2A_{21}}\Big)^{1/(\alpha_2+1)}r(x).$$

As $Z_1(m,n)$ is non-decreasing with respect to m and we are looking for non-negative quantities, it follows that r is a non-negative, non-decreasing and bounded function. Hence it admits a non-negative limit at infinity. Moreover from (53) we see that

$$\lim_{t\to\infty}\lim_{n\to\infty}\mathbf{E}\left[0^{Z_1((\alpha_2A_{21}t)^{\alpha_2/(1+\alpha_2)}g^*(n),n)}|\mathbf{Z}(n)\neq\mathbf{0}\right]=0$$

in view of $Z_1(m,n)+Z_2(m,n)\geq 1$. Hence $\lim_{x\to\infty}r(x)=(\alpha_2A_{21})^{1/(\alpha_2+1)}$ and

$$\lim_{t\to\infty}\lim_{n\to\infty}\mathbf{E}[s_1^{Z_1(tg^*(n),n)}|\mathbf{Z}(n)\neq\mathbf{0}]=1-(1-s_1)^{1/(1+\alpha_2)}.$$

From point (3) we also know that

$$\lim_{a\to 0}\lim_{n\to\infty}\mathbf{E}[s_1^{Z_1(an,n)}|\mathbf{Z}(n)\neq\mathbf{0}]=1-(1-s_1)^{1/(1+\alpha_2)}.$$

As $Z_1(m,n)$ is non-decreasing with respect to m, we end the proof of point (2) with (54).

Point (4): The proof is similar to that of point (3) and we give less details. In particular, the choice of s_2 has no influence on the limit, as there is no type 2 individuals anymore in the reduced process. Let $n-m=h(n)$ with $1\ll h(n)\ll n$. Set

$$1-s_1=1-\exp\left\{-\lambda_1\frac{Q_1(n)}{Q_1(h(n))}\right\}\sim\lambda_1\frac{Q_1(n)}{Q_1(h(n))}$$

and choose s_2 such that

$$(1-s_2)Q_{21}(h(n))\sim Q_{21}(\lambda_1^{-\alpha_1}n).$$

Then by (39), the branching property, (5) and (34) we conclude that

$$\mathbf{E}[1-\mathbf{s}^{\mathbf{Z}(m,n)}|\mathbf{Z}(n)\neq\mathbf{0}]=\frac{Q_{21}(n-h(n);\mathbf{1}-(\mathbf{1}-\mathbf{s})\otimes\mathbf{Q}_{21}(h(n)))}{Q_{21}(n)}$$
$$\sim\frac{Q_{21}(n-h(n);\mathbf{1}-\mathbf{Q}_{21}(\lambda_1^{-\alpha_1}n))}{Q_{21}(n)}=\frac{Q_{21}((1+\lambda_1^{-\alpha_1})n-h(n))}{Q_{21}(n)}\sim(1+\lambda_1^{-\alpha_1})^{-\frac{1}{\alpha_1(1+\alpha_2)}},$$

which ends the proof of Theorem 2.

7. Proof of Theorem 3

Throughout this section we assume that Conditions (2)-(4) and (10) are satisfied and that (s_1,s_2) belongs to $(0,1)^2$.

Point (0): Using (51), Condition (10) and (37) we obtain for $m\ll n$

$$\mathbf{E}\left[1-s_1^{Z_1(m,n)}\right]\leq A_{21}mQ_1(n-m)\ll nQ_1(n)\sim\sigma Q_2(n)\sim\sigma b^{-1}Q_{21}(n),$$

implying

$$0\leq\mathbf{E}\left[1-\mathbf{s}^{\mathbf{Z}(m,n)}\right]-\mathbf{E}\left[1-s_2^{Z_2(m,n)}\right]\leq\mathbf{E}\left[1-s_1^{Z_1(m,n)}\right]=o\left(Q_{21}(n)\right).$$

Further, by Lemma 2 and (37)

$$\mathbf{E}\left[1-s_2^{Z_2(m,n)}\right]=Q_2\left(m;1-(1-s_2)Q_{21}(n-m)\right)\sim(1-s_2)Q_{21}(n),$$

giving (22).

Point (1): The result is a direct consequence of (38), (39), and of the equivalence $Q_1(n)\sim\sigma Q_{21}(n)/b$ under Condition (10).

Point (2): Again it is a direct consequence of expressions (38) and (39). As it is very similar to the proofs of points (3) and (4) of Theorem 2, we do not give the details.

8. Convergence of finite dimensional distributions

In this section we study the limiting behavior of the finite-dimensional distributions of the properly scaled reduced process $\{\mathbf{Z}(m,n),0\leq m\leq n\}$. To simplify notation we let

$$J_i^{(m,n)}(\mathbf{s})=\mathbf{E}\left[\mathbf{s}^{\mathbf{Z}(m,n)}|\mathbf{Z}(n)\neq\mathbf{0},\mathbf{Z}(0)=\mathbf{e}_i\right],\quad i=1,2$$

and, given $0\leq k_0<k_1<...<k_p\leq n$ set $\mathbf{k}=(k_0,k_1,...,k_p)$. Putting $\mathbf{S}_l=(s_{1l},s_{2l})$ denote

$$\hat{J}_2^{(k_0,k_1,...,k_p,n)}(\mathbf{S}_1,...,\mathbf{S}_p)=\hat{J}_2^{(\mathbf{k},n)}(\mathbf{S}_1,...,\mathbf{S}_p)=\mathbf{E}\left[\prod_{l=1}^{p}\mathbf{S}_l^{\mathbf{Z}(k_l,n)}\middle|\mathbf{Z}(k_0,n)=\mathbf{e}_2\right],$$
$$\hat{J}_1^{(k_0,k_1,...,k_p,n)}(s_{11},...,s_{1p})=\hat{J}_1^{(\mathbf{k},n)}(s_{11},...,s_{1p})=\mathbf{E}\left[\prod_{l=1}^{p}s_{1l}^{Z_1(k_l,n)}\middle|\mathbf{Z}(k_0,n)=\mathbf{e}_1\right],$$

and

$$\hat{\mathbf{J}}^{(\mathbf{k},n)}(\mathbf{S}_1,...,\mathbf{S}_p)=\left(\hat{J}_1^{(\mathbf{k},n)}(s_{11},...,s_{1p}),\hat{J}_2^{(\mathbf{k},n)}(\mathbf{S}_1,...,\mathbf{S}_p)\right).$$

The next statement is a simple observation following from Corollary 2 in [22].

Lemma 7. *For any $0\leq k_0<k_1<...<k_p\leq n$ we have*

$$\hat{J}_2^{(\mathbf{k},n)}(\mathbf{S}_1,...,\mathbf{S}_p)=\hat{J}_2^{(k_0,k_1,n)}\left(\mathbf{S}_1\otimes\hat{\mathbf{J}}^{(k_1,k_2,...,k_p,n)}(\mathbf{S}_2,...,\mathbf{S}_p)\right)$$
$$=J_2^{(k_1-k_0,n-k_0)}\left(\mathbf{S}_1\otimes\mathbf{J}^{(k_2-k_1,n-k_1)}\left(\mathbf{S}_2\otimes\left(...\left(\mathbf{S}_{p-1}\otimes\mathbf{J}^{(k_p-k_{p-1},n-k_{p-1})}(\mathbf{S}_p)\right)...\right)\right)\right).$$

In particular, if $\mathbf{k}=(0,k_1,k_2)$, *then*

$$\hat{J}_2^{(0,k_1,k_2,n)}(\mathbf{S}_1,\mathbf{S}_2)=J_2^{(k_1,n)}(\mathbf{S}_1\otimes\mathbf{J}^{(k_2-k_1,n-k_1)}(\mathbf{S}_2)).$$

Remark 2. *It follows from Lemma 7 that the process* $\{\mathbf{Z}(m,n),0\le m\le n|\,\mathbf{Z}(n)\neq\mathbf{0}\}$ *is, for a given* n, *an inhomogeneous discrete-time Markov branching process in* m.

Observe that for $\mathbf{k}=(k_0,k_1)$

$$\mathbf{E}\Big[\mathbf{s}^{\mathbf{Z}(k_1,n)}|\mathbf{Z}(k_0,n)=(r_1,r_2)\Big]=\left(\hat{J}_1^{(\mathbf{k},n)}(s_1)\right)^{r_1}\left(\hat{J}_2^{(\mathbf{k},n)}(s_1,s_2)\right)^{r_2}. \tag{55}$$

Hence to investigate this quantity we can use the equalities

$$\hat{J}_1^{(\mathbf{k},n)}(s_1)=1-\frac{Q_1(k_1-k_0,1-(1-s_1)Q_1(n-k_1))}{Q_1(n-k_0)} \tag{56}$$

and

$$\hat{J}_2^{(\mathbf{k},n)}(\mathbf{s})=1-\frac{Q_{21}(k_1-k_0;\mathbf{1}-(\mathbf{1}-\mathbf{s})\otimes\mathbf{Q}_{21}(n-k_1))}{Q_{21}(n-k_0)}. \tag{57}$$

We are now able to prove convergence of the finite-dimensional distributions of the prelimiting processes to the corresponding finite-dimensional distributions of the limiting processes in Theorems 4 to 6.

8.1. **Convergence of finite-dimensional distributions in Theorem 4.**
Point (1): For $t\ge 0$ introduce the function

$$f_{21}(t;s_2)=1-\left(1-e^{-t}+e^{-t}(1-s_2)^{-\alpha_2}\right)^{-\frac{1}{\alpha_2}}=\lim_{n\to\infty}\mathbf{E}_2\left[\mathbf{s}^{\mathbf{Z}((1-e^{-t})n,n)}|\mathbf{Z}(n)\neq\mathbf{0}\right],$$

whose first order derivative with respect to t is

$$\frac{\partial f_{21}(t;s_2)}{\partial t}=\frac{1}{\alpha_2}\left(e^{-t}-e^{-t}(1-s_2)^{-\alpha_2}\right)\left(1-e^{-t}+e^{-t}(1-s_2)^{-\alpha_2}\right)^{-1-1/\alpha_1}.$$

Recalling the definition of $g_2^{(X)}$ in (23) we get

$$\begin{aligned}
g_2^{(X)}(f_{21}(t;s_2))-f_{21}(t;s_2)&=\frac{1}{\alpha_2}\Big((1-f_{21}(t;s_2))^{1+\alpha_2}-1+(1+\alpha_2)f_{21}(t;s_2)\Big)-f_{21}(t;s_2)\\
&=\frac{1}{\alpha_2}\Big((1-e^{-t}+e^{-t}(1-s_2)^{-\alpha_2})^{-1-1/\alpha_2}-(1-e^{-t}+e^{-t}(1-s_2)^{-\alpha_2})^{-1/\alpha_2}\Big)\\
&=\frac{1}{\alpha_2}(1-e^{-t}+e^{-t}(1-s_2)^{-\alpha_2})^{-1-1/\alpha_2}\Big(1-(1-e^{-t}+e^{-t}(1-s_2)^{-\alpha_2})\Big)=\frac{\partial f_{21}(t;s_2)}{\partial t}.
\end{aligned}$$

Applying Lemma 6 we conclude that for $s_1,s_2\in[0,1]$

$$\lim_{n\to\infty}J_2^{(0,(1-e^{-t})n,n)}(\mathbf{s})=f_{21}(t;s_2)=\mathbf{E}\left[s_2^{X(t)}|X(0)=1\right],$$

where $X(\cdot)$ has been specified in Definition 1. This, along with Theorem 1(1) proves convergence of one-dimensional distributions of the prelimiting process to that of $X(\cdot)$.

In view of Lemma 7, to check the needed convergence of arbitrary finite-dimensional distributions it is sufficient to consider the case $p=2$. Let

$$k_1=\left[(1-e^{-t_1})n\right]<\left[(1-e^{-t_2})n\right]=k_2,\quad 0<t_1<t_2.$$

By the previous estimates, Lemma 7, Theorem 1(1) and the uniform continuity in t_1 and t_2 of the functions involved, we get

$$\begin{aligned}
\lim_{n\to\infty}\hat{J}_2^{(\mathbf{k},n)}(\mathbf{S}_1,\mathbf{S}_2)&=\lim_{n\to\infty}J_2^{(\Delta_1,n)}(\mathbf{S}_1\otimes\mathbf{J}^{(\Delta_2,n-k_1)}(\mathbf{S}_2))=f_{21}\left(t_1;s_{21}\lim_{n\to\infty}J_2^{(\Delta_2,n-k_1)}(\mathbf{S}_2)\right)\\
&=f_{21}\left(t_1;s_{21}f_{21}\left(t_2-t_1;s_{22}\right)\right)=\mathbf{E}\left[s_{21}^{X(t_1)}s_{22}^{X(t_2)}|X(0)=1\right]
\end{aligned}$$

as desired.

Point (2): Convergence of one-dimensional distributions has been established in Theorem 1(2). In particular, there is no type 1 individuals in the limit process. As before, to check the needed convergence of arbitrary finite-dimensional distributions it is sufficient to consider the case $p=2$ and to calculate the transition density of the limiting process for type 2 individuals. Let h be an integer valued function such that $h^*(n) \ll h(n) \ll n$, and $k_0 = n - t_0 h(n) < k_1 = n - t_1 h(n)$. Take s_2 satisfying, for a positive λ

$$1 - s_2 = 1 - \exp\left\{-\lambda \frac{Q_2(n)}{Q_{21}(t_2 h(n))}\right\} \sim \lambda \frac{Q_2(n)}{Q_{21}(t_2 h(n))}.$$

Let, further,

$$Z_2(k_1, n) = y \frac{Q_{21}(t_1 h(n))}{Q_2(n)}$$

(recall that we agree to consider the equalities of the form $Z(x,n) = y$ as $Z([x], n) = [y]$). Then from (56) and the fact that there is no type 1 individuals we get for s_1 in $[0,1]$,

$$1 - \hat{J}_2^{(\mathbf{k},n)}(\mathbf{s}) \sim \frac{1 - \mathbf{E}_2\left[e^{-\lambda Q_2(n) Z_2((t_0 - t_1)h(n))}\right]}{Q_{21}(t_0 h(n))} \sim \frac{\lambda Q_2(n)}{Q_{21}(t_0 h(n))},$$

where we applied Lemma 2. Hence we deduce that

$$\log \mathbf{E}\left[\mathbf{s}^{\mathbf{Z}(k_1,n)} | \mathbf{Z}(k_0, n) = y \frac{Q_{21}(t_0 h(n))}{Q_2(n)} \mathbf{e}_2\right] = y \frac{Q_{21}(t_0 h(n))}{Q_2(n)} \log \hat{J}_2^{(\mathbf{k},n)}(\mathbf{s}) \sim -\lambda y.$$

Point (3): Once again, it is sufficient to calculate the transition density of the limit process only. Let $k_0 = n - t_0 h^*(n) < k_1 = n - t_1 h^*(n)$, where $h^*(n)$ has been defined in (12). Applying first (56) we get

$$\begin{aligned}
1 - \hat{J}_1^{(\mathbf{k},n)}(s_1) &\sim \frac{1 - \mathbf{E}_1\left[e^{-(1-s_1)Q_1(t_1 h^*(n)) Z_1((t_0 - t_1)h^*(n))}\right]}{Q_1(t_0 h^*(n))} \\
&\sim \frac{1 - \mathbf{E}_1\left[e^{-(1-s_1)((t_0 - t_1)/t_1)^{1/\alpha_1} Q_1((t_0 - t_1)h^*(n)) Z_1((t_0 - t_1)h^*(n))}\right]}{((t_0 - t_1)/t_0)^{1/\alpha_1} Q_1((t_0 - t_1)h^*(n))} \\
&\sim \left(\frac{t_0}{t_0 - t_1}\right)^{\frac{1}{\alpha_1}} \left[1 + \frac{t_1}{t_0 - t_1}(1 - s_1)^{-\alpha_1}\right]^{-\frac{1}{\alpha_1}} = \left(1 - \frac{t_1}{t_0} + \frac{t_1}{t_0}(1 - s_1)^{-\alpha_1}\right)^{-\frac{1}{\alpha_1}}.
\end{aligned}$$

Introduce now

$$s_2 = \exp\left\{-\lambda_2 \frac{Q_2(n)}{Q_{21}(t_1 h^*(n))}\right\}.$$

Then, by Markov property and (39) we get

$$\begin{aligned}
1 - \hat{J}_2^{(\mathbf{k},n)}(\mathbf{s}) &= 1 - \mathbf{E}_2\left[\mathbf{s}^{\mathbf{Z}((t_0 - t_1)h^*(n), t_0 h^*(n))} | \mathbf{Z}(t_0 h^*(n)) \neq \mathbf{0}\right] \\
&= \frac{Q_{21}((t_0 - t_1) h^*(n); \mathbf{1} - (\mathbf{1} - \mathbf{s}) \otimes \mathbf{Q}_{21}(t_1 h^*(n)))}{Q_{21}(t_0 h^*(n))} \\
&\sim \frac{Q_{21}((t_0 - t_1) h^*(n); 1 - (1 - s_1) t_1^{-1/\alpha_1} Q_1(h^*(n)), 1 - \lambda_2 Q_2(n))}{Q_{21}(t_0 h^*(n))} \\
&\sim \frac{Q_2(n)}{Q_{21}(t_0 h^*(n))}\left[(1 - s_1)^{1-\alpha_1} t_1^{1 - \frac{1}{\alpha_1}} \left(1 - \left(1 + \left(\frac{t_0}{t_1} - 1\right)(1 - s_1)^{\alpha_1}\right)^{1 - \frac{1}{\alpha_1}}\right) + \lambda_2\right],
\end{aligned}$$

where for the last equivalence we applied (45) with $a = t_0 - t_1$ and $\lambda_1 = (1-s_1)t_1^{-1/\alpha_1}$. Hence,

$$\mathbf{E}\left[s_1^{Z_1(k_1,n)} e^{-\lambda_2 \frac{Q_2(n)}{Q_{21}(x_1 h^*(n))} Z_2(k_1,n)} | \mathbf{Z}(k_0,n) = y \frac{Q_{21}(x_0 h^*(n))}{Q_2(n)} \mathbf{e}_2 \right]$$
$$\sim \exp\left\{ - \left((1-s_1)^{1-\alpha_1} t_1^{1-1/\alpha_1} \left(1 - \left(1 + \left(\frac{t_0}{t_1} - 1 \right) (1-s_1)^{\alpha_1} \right)^{1-1/\alpha_1} \right) + \lambda_2 \right) y \right\}.$$

This proves the desired convergence of the finite-dimensional distribution of the prelimiting process to those of $\mathbf{G}(\cdot)$.

Point (4): Let h be an integer valued function such that $1 \ll h(n) \ll h^*(n)$, and λ_1, λ_2 be positive numbers. Selecting $k_0 = n - t_0 h(n) < n - t_1 h(n) = k_1$, we get by using (57)

$$1 - \hat{J}_2^{(k_0,k_1,n)} \left(e^{-\lambda_1 \frac{Q_1(h^*(n))}{Q_1(t_1 h(n))}}, e^{-\lambda_2 \frac{Q_2(n)}{Q_{21}(t_1 h(n))}} \right) \sim \frac{1 - \mathbf{E}_2 \Big[\exp\{-\lambda_2 Q_2(n) Z_2((t_0 - t_1)h(n))\} \Big]}{Q_{21}(t_0 h(n))}$$
$$+ \frac{\mathbf{E}_{21} \Big[e^{-\lambda_2 Q_2(n) Z_2((t_0-t_1)h(n))} \Big(1 - e^{-\lambda_1 Q_1(h^*(n)) Z_1((t_0-t_1)h(n))} \Big) \Big]}{Q_{21}(t_0 h(n))}.$$

By applying twice Lemma 2 as $h(n) \ll h^*(n)$ and $h(n) \ll n$ we get that the first summand is equivalent to

$$\frac{\lambda_2 Q_2(n)}{Q_{21}(t_0 h(n))},$$

and the second summand is smaller than a term equivalent to

$$\frac{\lambda_1 Q_1(h^*(n))}{Q_{21}(t_0 h(n))} = o\Big(\frac{Q_2(n)}{Q_{21}(t_0 h(n))} \Big).$$

This yields

$$\mathbf{E}_2 \Big[\exp\left\{ -\lambda_1 \frac{Z_1(k_1,n) Q_1(h^*(n))}{Q_1(t_1 h(n))} - \lambda_2 \frac{Z_2(k_1,n) Q_2(n)}{Q_{21}(t_1 h(n))} \right\} | \mathbf{Z}(k_0,n) = y_2 \frac{Q_{21}(t_0 h(n))}{Q_2(n)} \mathbf{e}_2 \Big] \sim e^{-\lambda_2 y_2}.$$

In the same way we deduce that

$$1 - \mathbf{E}_2 \Big[\exp\left\{ -\lambda_1 \frac{Z_1(k_1,n) Q_1(h^*(n))}{Q_1(t_1 h(n))} \right\} | \mathbf{Z}(k_0,n) = y_1 \frac{Q_1(t_0 h(n))}{Q_1(h^*(n))} \mathbf{e}_1 \Big] \sim e^{-\lambda_1 y_1}.$$

The two last equivalences justify the claimed convergence of finite-dimensional distributions.

8.2. **Convergence of finite-dimensional distributions in Theorem 5.**

Point (1): Applying Theorem 2(1) we can consider the function

$$f_{21}(t;\mathbf{s}) := \lim_{n\to\infty} \mathbf{E}_2 \left[\mathbf{s}^{\mathbf{Z}(tg^*(n),n)} | \mathbf{Z}(n) \neq \mathbf{0} \right] = 1 - t^{-\frac{1}{\alpha_2}} \phi\Big((1-s_1) \frac{t^{\frac{1}{\alpha_2}+1}}{\alpha_2 A_{21}}, (1-s_2) t^{\frac{1}{\alpha_2}} \Big),$$

whose first order derivative with respect to t is

$$\begin{aligned} \frac{\partial f_{21}(t;\mathbf{s})}{\partial t} &= \frac{1}{\alpha_2} t^{-\frac{1}{\alpha_2}-1} \Big(\phi(\tilde{\lambda}_1, \tilde{\lambda}_2) - (1+\alpha_2) \tilde{\lambda}_1 \frac{\partial \phi}{\partial \lambda_1}(\tilde{\lambda}_1, \tilde{\lambda}_2) - \tilde{\lambda}_2 \frac{\partial \phi}{\partial \lambda_2}(\tilde{\lambda}_1, \tilde{\lambda}_2) \Big) \\ &= \frac{1}{\alpha_2} \Big(t^{-\frac{1}{\alpha_2}-1} \phi^{1+\alpha_2}(\tilde{\lambda}_1, \tilde{\lambda}_2) - 1 + s_1 \Big), \end{aligned}$$

where we have applied (15) and used the notations

$$\tilde{\lambda}_1 = (1-s_1) t^{1/\alpha_2+1} / (\alpha_2 A_{21}) \quad \text{and} \quad \tilde{\lambda}_2 = (1-s_2) t^{\frac{1}{\alpha_2}}.$$

Now taking $f_1(t;s_1) = s_1$ and recalling the definition of $g_{21}^{(Y)}$ in (24) we obtain

$$g_{21}^{(Y)}(s_1, f_{21}(t;\mathbf{s})) - f_{21}(t;\mathbf{s}) = \frac{\alpha_2}{1+\alpha_2} \frac{\partial f_{21}(t;\mathbf{s})}{\partial t}.$$

Applying Lemma 6 yields that f_{21} is the generating function of the process $\mathbf{Y}$ specified in Definition 1. Hence the desired convergence of finite-dimensional distributions follows by the same arguments as before.

Point (2): We take $g^*(n) \ll k_0 < k_1 \ll n$. From Theorem 2(2) we know that there is no more type 2 individuals in the limit process. Applying (56) and Lemma 2 we get

$$\hat{J}_1^{(\mathbf{k},n)}(s_1) = 1 - \frac{Q_1(k_1 - k_0, 1 - (1-s_1)Q_1(n-k_1))}{Q_1(n-k_0)} \sim 1 - \frac{(1-s_1)Q_1(n-k_1)}{Q_1(n-k_0)} \sim s_1.$$

Point (3): Again as there is no type 2 individuals it is sufficient to study the dynamics of type 1 individuals. Applying Theorem 2(3) we can consider the function

$$f_1(t;s_1) = \lim_{n\to\infty} \mathbf{E}_2\left[\mathbf{s}^{\mathbf{Z}((1-e^{-t})n,n)} | \mathbf{Z}(n) \neq \mathbf{0}\right] = 1 - ((1-e^{-t}) + e^{-t}(1-s_1)^{-\alpha_1})^{-1/\alpha_1(1+\alpha_2)},$$

whose first order derivative with respect to t is

$$\frac{\partial f_1(t;s_1)}{\partial t} = \frac{1}{\alpha_1(1+\alpha_2)}\left(e^{-t} - e^{-t}(1-s_1)^{-\alpha_1}\right)\left(1 - e^{-t} + e^{-t}(1-s_1)^{-\alpha_1}\right)^{-1-1/(\alpha_1(1+\alpha_2))}.$$

Recalling the definition of $g_1^{(V)}$ in (25) we obtain

$$g_1^{(V)}(f_1(t;s_1)) - f_1(t;s_1) = \frac{1}{\alpha_1(1+\alpha_2)}\Big((1-f_1(t;s_1))^{1+\alpha_1(1+\alpha_2)} - 1 + f_1(t;s_1)\Big) = \frac{\partial f_1(t;s_1)}{\partial t}.$$

Applying Lemma 6 yields that f_1 is the generating function of the process $V(\cdot)$ specified in Definition 1(3). Hence the desired convergence of finite-dimensional distributions follows.

Point (4): Let h be an integer-valued function satisfying $1 \ll h(n) \ll n$, and $k_0 = n - t_0 h(n) < k_1 = n - t_1 h(n)$. The one dimensional distribution of the limit process is given by Theorem 2(4). Applying Lemma 2 and (56) yields

$$1 - \mathbf{E}\Big[e^{-\lambda_1 \frac{Q_1(n)}{Q_1(t_1 h(n))} Z_1(k_1,n)} | \mathbf{Z}(k_0,n) = \mathbf{e}_1\Big] \sim \frac{1 - \mathbf{E}_1\left[e^{-\lambda_1 Q_1(n) Z_1((t_0-t_1)h(n))}\right]}{Q_1(t_0 h(n))} \sim \lambda_1 \frac{Q_1(n)}{Q_1(t_0 h(n))}.$$

Hence we deduce

$$\Big[e^{-\lambda_1 \frac{Q_1(n)}{Q_1(x_1 h(n))} Z_1(k_1,n)} | \mathbf{Z}(k_0,n) = y \frac{Q_1(x_0 h(n))}{Q_1(n)} \mathbf{e}_1\Big] \sim e^{-\lambda_1 y},$$

as required.

8.3. **Convergence of finite-dimensional distributions in Theorem 6.**

Point (1): Consider the function

$$\begin{aligned} 1 - f_1(t;s_1) &= 1 - \lim_{n\to\infty} \mathbf{E}_1\left[s_1^{Z_1((1-e^{-t})n,n)} | \mathbf{Z}(n) \neq \mathbf{0}\right] \\ &= \lim_{k_1\to\infty} \frac{1 - \mathbf{E}_1\left[e^{-(1-s_1)(e^t-1)^{1/\alpha_1} Q_1(k_1) Z_1(k_1)}\right]}{(1-e^{-t})^{1/\alpha_1} Q_1(k_1)} = \Big(1 - e^{-t} + e^{-t}(1-s_1)^{-\alpha_1}\Big)^{-\frac{1}{\alpha_1}}, \end{aligned}$$

where we applied (56) with $k_0 = 0$ and $k_1 = \left[(1-e^{-t})n\right]$, and (7). Then following the proof of Theorem 4(1) we obtain

$$g_1^{(W)}(f_1(t;s_1)) - f_1(t;s_1) = \frac{\partial f_1(t;s_1)}{\partial t}.$$

Now applying Theorem 3(1) we can consider the function

$$f_{21}(t;\mathbf{s}) := \lim_{n\to\infty} \mathbf{E}_2\left[\mathbf{s}^{\mathbf{Z}((1-e^{-t})n,n)} | \mathbf{Z}(n) \neq \mathbf{0}\right] = 1 - (1-e^{-t})^{-1/\alpha_2} \psi(\bar{\lambda}_1, \bar{\lambda}_2),$$

where, for the sake of readability we have introduced

$$\bar{\lambda}_1 = (1-s_1)(e^t-1)^{1/\alpha_1} \quad \text{and} \quad \bar{\lambda}_2 = (1-s_2)(e^t-1)^{1/\alpha_2}.$$

Since $1+\alpha_2^{-1}=\alpha_1^{-1}$ under Assumption (10), we get

$$\frac{\partial f_{21}(t;\mathbf{s})}{\partial t}=(1-e^{-t})^{-1/\alpha_2-1}\Big(\frac{e^{-t}}{\alpha_2}\psi(\bar\lambda_1,\bar\lambda_2)-\frac{\bar\lambda_1}{\alpha_1}\frac{\partial\psi}{\partial\bar\lambda_1}(\bar\lambda_1,\bar\lambda_2)-\frac{\lambda_2}{\alpha_2}\frac{\partial\psi}{\partial\bar\lambda_2}(\bar\lambda_1,\bar\lambda_2)\Big)$$
$$=\frac{(1-e^{-t})^{-1/\alpha_2-1}}{\alpha_2}\Big((e^{-t}-1)\psi(\bar\lambda_1,\bar\lambda_2)+b^{\alpha_2}\psi^{1+\alpha_2}(\bar\lambda_1,\bar\lambda_2)-\alpha_2A_{12}\bar\lambda_1\Big(1+\Big(\frac{b}{\sigma}\bar\lambda_1\Big)^{\alpha_1}\Big)^{-1/\alpha_1}\Big),$$

where we have applied (20). Further, recalling the definitions of $g_{21}^{(W)}$ and κ in (28) and (26) we see that

$$g_{21}^{(W)}(f_1(t;s_1),f_{21}(t;\mathbf{s}))-f_{21}(t;\mathbf{s})$$
$$=\frac{1}{\kappa}\Big(\frac{\sigma A_{12}}{b}f_1(t;s_1)+\frac{b^{\alpha_2}}{\alpha_2}\Big((1-f_{21}(t;\mathbf{s}))^{1+\alpha_2}-1+(1+\alpha_2)f_{21}(t;\mathbf{s})\Big)\Big)-f_{21}(t;\mathbf{s}).$$

Observing that

$$\frac{1}{\kappa}\frac{b^{\alpha_2}(1+\alpha_2)}{\alpha_2}-1=\frac{\alpha_2}{(1+\alpha_2)b^{\alpha_2}-1}\frac{b^{\alpha_2}(1+\alpha_2)}{\alpha_2}-1=\frac{1}{(1+\alpha_2)b^{\alpha_2}-1}=\frac{1}{\alpha_2\kappa}$$

and recalling the definition of b in (19) leads to

$$g_{21}^{(W)}(f_1(t;s_1),f_{21}(t;\mathbf{s}))-f_{21}(t;\mathbf{s})$$
$$=\frac{1}{\alpha_2\kappa}\Big(b^{\alpha_2}(1-f_{21}(t;\mathbf{s}))^{1+\alpha_2}-\frac{\sigma\alpha_2A_{12}}{b}(1-f_1(t;s_1))-1+f_{21}(t;\mathbf{s})\Big)$$
$$=\frac{(1-e^{-t})^{-1-1/\alpha_2}}{\alpha_2\kappa}\Big(b^{\alpha_2}\psi^{1+\alpha_2}(\bar\lambda_1,\bar\lambda_2)-\frac{\sigma\alpha_2A_{12}}{b}\Big(1+\frac{e^{-t}}{1-e^{-t}}(1-s_1)^{-\alpha_1}\Big)^{-\frac{1}{\alpha_1}}-(1-e^{-t})\psi(\bar\lambda_1,\bar\lambda_2)\Big),$$

where we have used again the equality $1+\alpha_2^{-1}=\alpha_1^{-1}$. This, finally, yields

$$g_{21}^{(W)}(f_1(t;s_1),f_{21}(t;\mathbf{s}))-f_{21}(t;\mathbf{s})=\frac{1}{\kappa}\frac{\partial f_{21}(t;\mathbf{s})}{\partial t}.$$

Applying Lemma 6 we conclude that f_{21} is the generating function of the process $\mathbf{W}(\cdot)$ specified in Definition 1(4).

Point (2): Let $1\ll h(n)\ll n$ and $k_0=n-t_0h(n)<k_1=n-t_1h(n)$ for $0<t_1<t_0$. Choose $\lambda_1,\lambda_2\geq 0$. Using (56), the fact that $h(n)\gg 1$ and recalling Lemma 2 we see that

$$1-\hat J_1^{(k_0,k_1,n)}\Big(e^{-\lambda_1\frac{Q_1(n)}{Q_1(x_1h(n))}}\Big)\sim\frac{Q_1((t_0-t_1)h(n),1-\lambda_1Q_1(n))}{Q_1(x_0h(n))}\sim\frac{\lambda_1Q_1(n)}{Q_1(x_0h(n))}.$$

Hence

$$\log\mathbf{E}\Big[e^{-\lambda_1\frac{Q_1(n)}{Q_1(t_1h(n))}Z_1(k_1,n)}|\mathbf{Z}(k_0,n)=y\frac{Q_1(t_0h(n))}{Q_1(n)}\mathbf{e}_1\Big]\sim-\lambda_1y.$$

Then from (57) we obtain

$$1-\hat J_2^{(k_0,k_1,n)}\Big(e^{-\frac{\lambda_1Q_1(n)}{Q_1(t_1h(n))}},e^{-\frac{\lambda_2Q_2(n)}{Q_2(t_1h(n))}}\Big)\sim\frac{1-\mathbf{E}_2\Big[e^{-\lambda_1Q_1(n)Z_1((t_0-t_1)h(n))}e^{-\lambda_2Q_{21}(n)Z_2((t_0-t_1)h(n))}\Big]}{Q_{21}(t_0h(n))}$$
$$=\frac{1-\mathbf{E}_2\Big[e^{-\lambda_2Q_{21}(n)Z_2((t_0-t_1)h(n))}\Big]}{Q_{21}(t_0h(n))}+\frac{\mathbf{E}_2\Big[e^{-\lambda_2Q_{21}(n)Z_2((t_0-t_1)h(n))}\Big(1-e^{-\lambda_1Q_1(n)Z_1((t_0-t_1)h(n))}\Big)\Big]}{Q_{21}(t_0h(n))}.$$

To control the first term we apply Lemma 2 and Equivalence (37):

$$\frac{1-\mathbf{E}_2\Big[e^{-\lambda_2Q_{21}(n)Z_2((t_0-t_1)h(n))}\Big]}{Q_{21}(t_0h(n))}\sim\frac{\lambda_2Q_{21}(n)}{Q_{21}(t_0h(n))}.$$

The second term is non-negative and bounded by

$$\frac{\mathbf{E}_2\left[1-e^{-\lambda_1 Q_1(n) Z_1((t_0-t_1)h(n))}\right]}{Q_{21}(t_0 h(n))} \sim \frac{\lambda_1(t_0-t_1)h(n)Q_1(n)}{Q_{21}(t_0 h(n))} = o\Big(\frac{\lambda_2 Q_{21}(n)}{Q_{21}(t_0 h(n))}\Big),$$

where the equality is a direct consequence of Condition (10) and Equivalence (37). This ends the proof of point (2).

9. Tightness

In this section we complete the proof of Theorems 4-6 by checking tightness. A key tool will be a slightly simplified version of Theorem 6.5.4 in [9] giving a convergence criterion in Skorokhod topology for a class of Markov processes which we formulate to simplify references. Let $a_1 \leq a_2$ be in $\mathbf{R}_+$ and

$$\{\mathbf{K}_n(u) = (K_{1n}(u), K_{2n}(u)), a_1 \leq u \leq a_2\}, \quad n = 1, 2, ...$$

be a sequence of Markov processes with values in $\mathbf{R}^2$ whose trajectories belong with probability 1 to the space $\mathbf{D}_{[a_1,a_2]}(\mathbf{R}^2)$ of càdlàg functions on $[a_1, a_2]$.

Theorem 9. *If the finite-dimensional distributions of* $\{\mathbf{K}_n(u), a_1 \leq u \leq a_2\}$ *converge, as* $n \to \infty$ *to the respective finite-dimensional distributions of a process* $\{\mathbf{K}(u), a_1 \leq u \leq a_2\}$ *and for any* $\varepsilon > 0$

$$\lim_{g\downarrow 0} \overline{\lim_{n\to\infty}} \sup_{0\leq u_1-u_0\leq g} \sup_{\mathbf{y}\in\mathbb{R}^2} \mathbf{P}\left(\|\mathbf{K}_n(u_1)-\mathbf{y}\| > \varepsilon | \, \mathbf{K}_n(u_0)=\mathbf{y}\right) = 0,$$

then, as $n \to \infty$

$$\mathcal{L}\{\mathbf{K}_n(u), a_1 \leq u \leq a_2\} \Longrightarrow \mathcal{L}\{\mathbf{K}(u), a_1 \leq u \leq a_2\}.$$

With this theorem in hands, the method of proving the desired tightness is, essentially, the same for all three theorems and has been used in various situations in [8], [19] and [21]. For instance, to prove the tightness in Theorems 4 and 5 one can apply a combination of arguments used in [8] and [19]. The main difference with these previous proofs is related with the non finiteness of offspring variances under Conditions (2)-(4). Since the remaining parts of the proofs follow the reasoning used in [8] and [19] we check in this section the tightness for Theorem 6 only. In what follows it will be convenient to write $\mathbf{P}^{(n)}(\mathcal{B})$ for $\mathbf{P}(\mathcal{B}|\mathbf{Z}(n) \neq \mathbf{0})$ for any admissible event $\mathcal{B}$ and to use the notation $\|\mathbf{x}\| = |x_1| + |x_2|$ for $\mathbf{x} = (x_1, x_2)$.

Proof of tightness for Theorem 6 (1) . We will use in this proof ideas from [8]. As we have mentioned, according to Lemma 7 the law $\mathbf{P}(\{\mathbf{Z}(m,n), 0 \leq m \leq n\} \in (\cdot)|\mathbf{Z}(n) \neq \mathbf{0})$ specifies, for each fixed n an inhomogeneous Markov branching process. We denote its transition probabilities by $\mathbf{P}^{(n)}(m_1, \mathbf{z}; m_2, (\cdot))$. In the case under consideration, $\mathbf{K}_n$ writes:

$$\mathbf{K}_n(u) = \{\mathbf{Z}(un, n)|\mathbf{Z}(n) \neq \mathbf{0}\}, 0 \leq u < 1.$$

Hence, we will use Theorem 9 for each $[0, U] \subset [0, 1)$. Denote

$$\mathcal{C}(k) = \left\{\mathbf{z} \in \mathbb{Z}_+^2 : \|\mathbf{z}\| \leq k\right\},$$

take $0 < u_0 < u_1 < 1$, set

$$m_j = u_j n = \left(1 - e^{-t_j}\right) n, \ j = 0, 1$$

and, finally, let

$$\mathcal{U}(g) = \{(u_1, u_0) \in [0, U] \times [0, U] : 0 \leq u_1 - u_0 \leq g\}.$$

Lemma 8. *Under the conditions of Theorem 6(1) for any fixed* k *and* $0 < U < 1$

$$\lim_{g\downarrow 0} \overline{\lim_{n\to\infty}} \sup_{\mathcal{U}(g)} \sup_{\mathbf{z}\in\mathcal{C}(k)} \mathbf{P}^{(n)}(\mathbf{Z}(m_1; n) \neq \mathbf{z}|\mathbf{Z}(m_0; n) = \mathbf{z}) = 0.$$

Proof. Let n be fixed in $\mathbf{N}$. Notice that when a jump occurs in the process $\{\mathbf{Z}(m,n), 0 \leq m \leq n\}$, the process $|\mathbf{Z}(.,n)| + Z_1(.,n)$ is incremented by one. Hence if $m_0 \leq m_1 \leq m_2$, then $\mathbf{Z}(m_0,n) \neq \mathbf{Z}(m_1,n)$ implies $\mathbf{Z}(m_0,n) \neq \mathbf{Z}(m_2,n)$. Adding Markov property we get

$$\begin{aligned}
&\sup_{\mathcal{U}(g)} \sup_{\mathbf{z}\in\mathcal{C}(k)} \mathbf{P}^{(n)}(\mathbf{Z}(m_1;n) \neq \mathbf{z}|\mathbf{Z}(m_0;n) = \mathbf{z}) \\
&\qquad \leq \sup_{\mathcal{U}(g)} \sup_{\mathbf{z}\in\mathcal{C}(k)} \mathbf{P}^{(n)}(\mathbf{Z}(m_0+gn;n) \neq \mathbf{z}|\mathbf{Z}(m_0;n) = \mathbf{z}) \\
&\qquad = \sup_{u_0\in[0,U]} \sup_{\mathbf{z}\in\mathcal{C}(k)} \mathbf{P}^{(n-m_0)}(\mathbf{Z}(gn;n-m_0) \neq \mathbf{z}|\mathbf{Z}(0;n-m_0) = \mathbf{z}).
\end{aligned}$$

According to Lemma 7 and (7)

$$\begin{aligned}
1 - \lim_{n\to\infty} \mathbf{E}_1\left[s_1^{Z_1(gn,n-m_0)}|\mathbf{Z}(n-m_0) \neq \mathbf{0}\right] &= \lim_{n\to\infty} \frac{Q_1(gn;1-(1-s_1)Q_1(n-m_0-gn))}{Q_1(n-m_0)} \\
&= (1-u_0)^{1/\alpha_1}\Big(g + (1-s_1)^{-\alpha_1}(1-u_0-g)\Big)^{-1/\alpha_1}.
\end{aligned}$$

In particular,

$$\lim_{n\to\infty} \mathbf{P}^{(n-m_0)}(0,\mathbf{e}_1;gn,\{\mathbf{e}_1\}) = 1 - g/(1-u_0).$$

Therefore, for any $\varepsilon > 0$ and $(u_0,u_1) \in \mathcal{U}(g)$ there exists n_0 such that for all $n \geq n_0$

$$\lim_{n\to\infty} \mathbf{P}^{(n)}(m_0,\mathbf{e}_1;m_1,\{\mathbf{e}_1\}) \geq 1 - g/(1-u_0) - \varepsilon \geq 1 - g/(1-U) - \varepsilon = 1 - C_1 g - \varepsilon, \tag{58}$$

and

$$\lim_{g\downarrow 0} \lim_{n\to\infty} \sup_{\mathcal{U}(g)} \mathbf{P}^{(n)}(m_0,\mathbf{e}_1;m_1,\{\mathbf{e}_1\}) = 1. \tag{59}$$

Similarly, by Lemma 7 and Theorem 3(1) with $a = g/(1-u_0)$ we get

$$\begin{aligned}
\lim_{n\to\infty} \mathbf{E}\left[\mathbf{s}^{\mathbf{Z}(m_0+gn,n)}|\mathbf{Z}(m_0,n) = \mathbf{e}_2\right] &= \lim_{n\to\infty} \mathbf{E}_2\left[\mathbf{s}^{\mathbf{Z}(gn,n-m_0)}|\mathbf{Z}(n-m_0) \neq \mathbf{0}\right] \\
&= 1 - a^{-1/\alpha_2}\psi\left((1-s_1)\frac{\sigma}{b}\left(\frac{a}{1-a}\right)^{1/\alpha_1}, (1-s_2)\left(\frac{a}{1-a}\right)^{1/\alpha_2}\right).
\end{aligned}$$

Further, taking into account the boundary conditions (21) and that $\alpha_1^{-1} = 1 + \alpha_2^{-1}$ yields

$$\lim_{a\downarrow 0} a^{-1/\alpha_2}\psi\left((1-s_1)\frac{\sigma}{b}\left(\frac{a}{1-a}\right)^{1/\alpha_1}, (1-s_2)\left(\frac{a}{1-a}\right)^{1/\alpha_2}\right) = 1 - s_2.$$

Besides, for

$$\lambda_1^* = \frac{\sigma}{b}\left(\frac{a}{1-a}\right)^{1/\alpha_1} \quad \text{and} \quad \lambda_2^* = \left(\frac{a}{1-a}\right)^{1/\alpha_2}$$

we have

$$\lim_{n\to\infty} \mathbf{P}^{(n-m_0)}(0,\mathbf{e}_2;gn,\{\mathbf{e}_2\}) = (1-a)^{-1/\alpha_2} \frac{\partial\psi(\lambda_1,\lambda_2)}{\partial\lambda_2}\Big|_{(\lambda_1,\lambda_2)=(\lambda_1^*,\lambda_2^*)}.$$

Note that in view of (21)

$$\frac{\partial\psi(\lambda_1,\lambda_2)}{\partial\lambda_2}\Big|_{(\lambda_1,\lambda_2)=(\lambda_1^*,\lambda_2^*)} \geq 1 - C_2(g),$$

where $C_2(g) \to 0$ as $g \downarrow 0$. This implies, in particular, that for any $\varepsilon > 0$ there exists n_0 such that, for all $n \geq n_0$

$$\mathbf{P}^{(n)}(m_0,\mathbf{e}_2;m_1,\{\mathbf{e}_2\}) \geq (1-a)^{-1/\alpha_2}(1 - C_2(g)) - \varepsilon = 1 - C_3(g) - \varepsilon \tag{60}$$

where $C_3(g) \to 0$ as $g \downarrow 0$. In particular,

$$\lim_{g\downarrow 0} \lim_{n\to\infty} \sup_{\mathcal{U}(g)} \mathbf{P}^{(n)}(m_0, \mathbf{e}_2; m_1, \{\mathbf{e}_2\}) = 1. \tag{61}$$

Using (59)-(61), the branching property, the decomposability of the process and the positivity of the offspring number of each particle of the reduced process, we have for all $m_1 = u_1 n \geq m_0 = u_0 n$ and $\mathbf{z} \in \mathcal{C}(k)$,

$$\begin{aligned}
\lim_{g\downarrow 0} \liminf_{n\to\infty} \inf_{\mathcal{U}(g)} \mathbf{P}^{(n)}(m_0, \mathbf{z}; m_1, \{\mathbf{z}\}) &= \lim_{u_1\downarrow u_0} \liminf_{n\to\infty} \inf_{\mathcal{U}(g)} \prod_{j=1}^{2} (\mathbf{P}^{(n)}(m_0, \mathbf{e}_j; m_1, \{\mathbf{e}_j\}))^{z_j} \\
&\geq \lim_{g\downarrow 0} \liminf_{n\to\infty} \inf_{\mathcal{U}(g)} \prod_{j=1}^{2} (\mathbf{P}^{(n)}(m_0, \mathbf{e}_j; m_1, \{\mathbf{e}_j\}))^{k} = 1.
\end{aligned}$$

This implies the claim of the lemma. ■

Lemma 9. *Let the conditions of Theorem 6(1) be valid. If $m_j = u_j n,\ j = 0, 1, 2$ and $0 \leq u_0 < u_1 < u_2 < 1$ with $u_1 - u_0 \leq g$, then there exists $C_4(g) \to 0$ as $g \downarrow 0$ such that for any $\varepsilon > 0$ there exists n_0 such that for all $n \geq n_0$ and $\mathbf{z} \in \mathcal{C}(k)$,*

$$\mathbf{P}^{(n)}(\mathbf{Z}(m_1, n) = \mathbf{z} \,|\, \mathbf{Z}(m_0, n) = \mathbf{z}; \|\mathbf{Z}(m_2, n)\| \leq k) \geq \frac{\mathbf{P}^{(n)}(m_0, \mathbf{z}; m_1, \{\mathbf{z}\})\mathbf{P}^{(n)}(m_1, \mathbf{z}; m_2, \mathcal{C}(k))}{\mathbf{P}^{(n)}(m_1, \mathbf{z}; m_2, \mathcal{C}(k)) + (C_4(g) + 2\varepsilon)\, k}.$$

Proof. We have

$$\mathbf{P}^{(n)}(\mathbf{Z}(m_1, n) = \mathbf{z} \,|\mathbf{Z}(m_0, n) = \mathbf{z}, \|\mathbf{Z}(m_2, n)\| \leq k) = \mathbf{P}^{(n)}(m_0, \mathbf{z}; m_1, \{\mathbf{z}\}) \frac{\mathbf{P}^{(n)}(m_1, \mathbf{z}; m_2, \mathcal{C}(k))}{\mathbf{P}^{(n)}(m_0, \mathbf{z}; m_2, \mathcal{C}(k))}.$$

In view of (58)-(60), there exists n_0 such that for $n \geq n_0$,

$$\begin{aligned}
\mathbf{P}^{(n)}(m_0, \mathbf{z}; m_2, \mathcal{C}(k)) &= \sum_{\mathbf{z}'} \mathbf{P}^{(n)}(m_0, \mathbf{z}; m_1, \{\mathbf{z}'\})\mathbf{P}^{(n)}(m_1, \mathbf{z}'; m_2, \mathcal{C}(k)) \\
&\leq 1 - \mathbf{P}^{(n)}(m_0, \mathbf{z}; m_1, \{\mathbf{z}\}) + \mathbf{P}^{(n)}(m_1, \mathbf{z}; m_2, \mathcal{C}(k)) \\
&\leq 1 - (1 - C_3(g) - \varepsilon)^k (1 - C_1 g - \varepsilon)^k + \mathbf{P}^{(n)}(m_1, \mathbf{z}; m_2, \mathcal{C}(k)) \\
&\leq (C_1 g + C_3(g) + 2\varepsilon)\, k + \mathbf{P}^{(n)}(m_1, \mathbf{z}; m_2, \mathcal{C}(k)).
\end{aligned}$$

Hence the needed statement follows. ■

Let $U \in (0, 1)$ and $\delta \in (0, U)$ be fixed and, in addition, $U + \delta < 1$.

Lemma 10. *Under the conditions of Theorem 6(1) for any fixed k and $m_0 = u_0 n,\ m_1 = u_1 n,\ m_2 = (U + \delta)\, n$ we have*

$$\lim_{g\downarrow 0} \overline{\lim_{n\to\infty}} \sup_{\mathcal{U}(g)} \sup_{\mathbf{z}\in\mathcal{C}(k)} \mathbf{P}^{(n)}(\mathbf{Z}(m_1; n) \neq \mathbf{z} \,|\, \mathbf{Z}(m_0; n) = \mathbf{z}, \|\mathbf{Z}(m_2, n)\| \leq k) = 0.$$

Proof. Let us first notice that from (55) we can deduce that if $\mathbf{w} = (w_1, w_2) \leq \mathbf{z} = (z_1, z_2)$ (where the inequality is understood componentwise) then,

$$\begin{aligned}
\mathbf{P}^{(n)}(m_0, \mathbf{w}; m_1, \{\mathbf{w}\}) &= \left(\frac{\hat{J}_1^{(m_0,m_1,n)}(s_1)}{\partial s_1} |_{s_1=0} \right)^{w_1} \left(\frac{\hat{J}_2^{(m_0,m_1,n)}(\mathbf{s})}{\partial s_2} |_{\mathbf{s}=\mathbf{0}} \right)^{w_2} \\
&\geq \left(\frac{\hat{J}_1^{(m_0,m_1,n)}(s_1)}{\partial s_1} |_{s_1=0} \right)^{z_1} \left(\frac{\hat{J}_2^{(m_0,m_1,n)}(\mathbf{s})}{\partial s_2} |_{\mathbf{s}=\mathbf{0}} \right)^{z_2} = \mathbf{P}^{(n)}(m_0, \mathbf{z}; m_1, \{\mathbf{z}\}). \quad (62)
\end{aligned}$$

By Lemma 9, Equations (58) and (60), for $m_0 = u_0 n$, $m_1 = u_1 n$ and $\mathbf{z} \in \mathcal{C}(k)$

$$\begin{aligned}
&\mathbf{P}^{(n)}(\mathbf{Z}(m_1, n) = \mathbf{z} | \mathbf{Z}(m_0, n) = \mathbf{z}; \|\mathbf{Z}(m_2, n)\| \le k) \\
&\qquad \ge (1 - C_2(g) - \varepsilon)^k (1 - C_1 g - \varepsilon)^k \frac{\mathbf{P}^{(n)}(m_1, \mathbf{z}; m_2, \mathcal{C}(k))}{\mathbf{P}^{(n)}(m_1, \mathbf{z}; m_2, \mathcal{C}(k)) + (C_4(g) + 2\varepsilon) k}.
\end{aligned}$$

Using the decomposability hypothesis, the Markov property of the reduced processes, the inequality $m_2 - m_1 \le (U + \delta) n$ and (62) we obtain

$$\begin{aligned}
\mathbf{P}^{(n)}(m_1, \mathbf{z}; m_2, \mathcal{C}(k)) &\ge \mathbf{P}^{(n)}(m_1, \mathbf{z}; m_2, \{\mathbf{z}\}) = \prod_{j=1}^{2} (\mathbf{P}^{(n)}(m_1, \mathbf{e}_j; m_2, \{\mathbf{e}_j\}))^{z_j} \\
&\ge \prod_{j=1}^{2} (\mathbf{P}^{(n)}(m_1, \mathbf{e}_j; m_2, \{\mathbf{e}_j\}))^k \ge \prod_{j=1}^{2} \left(\mathbf{P}^{(n)}(0, \mathbf{e}_j; (U + \delta) n, \{\mathbf{e}_j\})\right)^k.
\end{aligned}$$

It follows from Subsection 8.3 that

$$\lim_{n\to\infty} \prod_{j=1}^{2} \left(\mathbf{P}^{(n)}(0, \mathbf{e}_j; (U + \delta) n, \{\mathbf{e}_j\})\right)^k = \prod_{j=1}^{2} \mathbf{P}^k(\mathbf{W}(-\log(1 - U - \delta)) = \mathbf{e}_j | \mathbf{W}(0) = \mathbf{e}_j) = B > 0.$$

Hence we get

$$\begin{aligned}
&\varliminf_{n\to\infty} \inf_{\mathcal{U}(g)} \inf_{\mathbf{z} \in \mathcal{C}(k)} \mathbf{P}_n(\mathbf{Z}(m_1; n) = \mathbf{z} | \mathbf{Z}(m_0; n) = \mathbf{z}, \|\mathbf{Z}(m_2, n)\| \le k) \\
&\qquad \ge (1 - C_2(g) - \varepsilon)^k (1 - C_1 g - \varepsilon)^k \frac{B}{B + (C_4(g) + 2\varepsilon) k}.
\end{aligned}$$

Letting first $g \downarrow 0$ and then $\varepsilon \downarrow 0$ completes the proof of the lemma. ■

Combining the statement of Lemma 10 and Theorem 9 about tightness and the finite dimensional convergence established in Section 8.3 point (1), we obtain the following statement:

Corollary 1. *Under the conditions of Lemma 10*

$$\begin{aligned}
&\mathcal{L}\left\{\mathbf{Z}(un, n), 0 \le u \le U \,\middle|\, \|\mathbf{Z}(m_2, n)\| \le k, \mathbf{Z}(n) \ne \mathbf{0}\right\} \\
&\qquad \Longrightarrow \mathcal{L}_{(0,1)} \left\{\mathbf{W}(u), 0 \le u \le U | \; \|\mathbf{W}(U + \delta)\| \le k\right\}.
\end{aligned}$$

Final steps in proving Theorem 6(1): Let for $0 < U < U_1 = U + \delta < 1$

$$\begin{aligned}
\mathbf{P}^{(n)}(U; (\cdot)) &= \mathbf{P}^{(n)}(\{\mathbf{Z}(un, n), 0 \le u \le U\} \in (\cdot)), \\
\mathbf{P}^{(n,k)}(U, U_1; (\cdot)) &= \mathbf{P}^{(n)}(\{\mathbf{Z}(un, n), 0 \le u \le U\} \in (\cdot) | \, \|\mathbf{Z}(U_1 n, n)\| \le k), \\
\bar{\mathbf{P}}^{(n,k)}(U, U_1; (\cdot)) &= \mathbf{P}^{(n)}(\{\mathbf{Z}(un, n), 0 \le u \le U\} \in (\cdot) | \, \|\mathbf{Z}(U_1 n, n)\| > k) \\
\mathcal{P}(U; (\cdot)) &= \mathbf{P}(\{\mathbf{W}(u), 0 \le u \le U\} \in (\cdot)), \\
\mathcal{P}^{(k)}(U, U_1; (\cdot)) &= \mathbf{P}(\{\mathbf{W}(u), 0 \le u \le U\} \in (\cdot) | \, \|\mathbf{W}(U_1)\| \le k).
\end{aligned}$$

Then, for a continuous real function ψ on $\mathbf{D}_{[0,U]}(\mathbf{Z}_+^2)$ such that $|\psi| \le q$ for a positive q we have

$$\begin{aligned}
\int \psi(x) \mathbf{P}^{(n)}(U; dx) &= \mathbf{P}^{(n)}(\|\mathbf{Z}(U_1 n, n)\| > k) \int \psi(x) \bar{\mathbf{P}}^{(n,k)}(U, U_1; dx) \\
&\quad + \mathbf{P}^{(n)}(\|\mathbf{Z}(U_1 n, n)\| \le k) \int \psi(x) \mathbf{P}^{(n,k)}(U, U_1; dx).
\end{aligned}$$

For the first summand we get

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbf{P}^{(n)}(\|\mathbf{Z}(U_1 n, n)\| > k) \int \psi(x) \bar{\mathbf{P}}^{(n,k)}(U, U_1; dx) \\
&\qquad \le q \limsup_{n\to\infty} \mathbf{P}^{(n)}(\|\mathbf{Z}(U_1 n, n)\| > k) = q \mathbf{P}(\|\mathbf{W}(U_1)\| > k) = o(1)
\end{aligned}$$

as $k \to \infty$ by the properties of $\mathbf{W}(\cdot)$. On the other hand, letting first $n \to \infty$ and then $k \to \infty$ we obtain from Lemma 10 and Theorem 9,

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{n\to\infty}\mathbf{P}_n(\|\mathbf{Z}(U_1 n,n)\|\le k)\int\psi(x)\mathbf{P}^{(k)}(U,U_1;dx)\\
&\qquad=\lim_{k\to\infty}\mathbf{P}(0<\|\mathbf{W}(U_1)\|\le k)\int\psi(x)\mathcal{P}^{(k)}(U,U_1;dx)\\
&\qquad\qquad=\lim_{k\to\infty}\int_{\{0<\|\mathbf{W}(U_1)\|\le k\}}\psi(x)\mathcal{P}(U,U_1;dx)=\int\psi(x)\mathcal{P}(U;dx).
\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\int\psi(x)\mathbf{P}^{(n)}(\mathbf{Z}((\cdot)n,n)\in dx)=\int\psi(x)\mathcal{P}(U;dx)$$

for any bounded continuous function on $\mathbf{D}_{[0,U]}(\mathbf{Z}_+^2)$ proving Theorem 6(1).

Proof of tightness for Theorem 6 (2). For $0<t<\infty$ and $1\ll h(n)\ll n$ let

$$\hat{\mathbf{Z}}(n;t)=\left(\hat{Z}_1(n;t),\hat{Z}_2(n;t)\right)=\left(\frac{Q_{21}(n)}{nQ_1(th(n))}Z_1(n-th(n),n),\frac{Q_2(n)}{Q_2(th(n))}Z_2(n-th(n),n)\right).$$

For $0<t_1<t_0$ we write

$$Z_1(n-t_1h(n),n)=Z_1^{(1)}(n-t_1h(n),n;t_0)+Z_1^{(2)}(n-t_1h(n),n;t_0)$$

where $Z_1^{(i)}(n-t_1h(n),n;t_0), i=1,2$ is the number of type 1 particles in the reduced process at moment $n-t_1h(n)$ with type i ancestors at time $n-t_0h(n)$. Set

$$\hat{\mathbf{z}}_n=(\hat{z}_{n1},\hat{z}_{n2})=\left(\frac{Q_{21}(n)}{nQ_1(t_0h(n))}z_1,\frac{Q_2(n)}{Q_2(t_0h(n))}z_2\right),\quad r_{ni}=\frac{Q_i(t_1h(n))}{Q_i(t_0h(n))},\quad i\in\{1,2\}$$

and, for $\varepsilon>0$

$$\Lambda_{n1}(t)=\varepsilon nQ_1(th(n))/Q_{21}(n),\quad \Lambda_{n2}(t)=\varepsilon Q_2(th(n))/Q_2(n).$$

Note that under our conditions $\Lambda_{ni}(t)\to\infty$ as $n\to\infty$. With these notations we have

$$\begin{aligned}
&\left\{\left\|\hat{\mathbf{Z}}(n;t_1)-\hat{\mathbf{z}}_n\right\|>3\varepsilon\right\}\subset\left\{|Z_2(n-t_1h(n),n)-r_{n2}z_2|>\Lambda_{n2}(t_1)\right\}\\
&\qquad\cup\left\{\left|Z_1^{(1)}(n-t_1h(n),n;t_0)-r_{n1}z_1\right|>\Lambda_{n1}(t_1)\right\}\cup\left\{Z_1^{(2)}(n-t_1h(n),n;t_0)>\Lambda_{n1}(t_1)\right\}.
\end{aligned}$$

To go further we need the representation

$$\hat{Z}_2(n;t_1)=\frac{Q_2(n)}{Q_2(t_1h(n))}\sum_{k=1}^{Z_2(n-t_0h(n),n)}\zeta_k(n;t_0,t_1),$$

where $\zeta_k(n;t_0,t_1)\overset{d}{=}\zeta(n;t_0,t_1)$ and $\zeta(n;t_0,t_1)$ is the number of type 2 particles in the reduced process at time $n-t_1h(n)$ being descendants of a particle of the reduced process existing at time $n-t_0h(n)$. Observe that $\zeta(n;t_0,t_1)\le Z_2(n-t_1h(n))$ and $\mathbf{E}Z_2^{\gamma}(n-t_1h(n))<\infty$ for any $\gamma\in[1,1+\alpha_2)$ in view of Condition (3) and Theorem 2.3.7 in [12]. Notice that

$$\begin{aligned}
\mathbf{E}\left[\zeta(n;t_0,t_1)|Z_2(n-t_0h(n),n)=1\right]&=\mathbf{E}_2\left[Z_2((t_0-t_1)\,h(n),t_0h(n))|Z_2(t_0h(n))>0\right]\\
&=Q_2(t_1h(n))/Q_2(t_0h(n))
\end{aligned}$$

and that $\{\zeta_k(n;t_0,t_1),\ k=1,2,...,Z_2(n-t_0h(n),n)\}$ are iid given $Z_2(n-t_0h(n),n)$. Using von Bahr-Esseen inequality ([2] Theorem 2) for $\gamma\in(1,1+\alpha_2)$, Markov branching property and

Lemma 7, we obtain that

$$\begin{aligned}
(\Lambda_{n2}(t_1))^{\gamma}\,\mathbf{P}^{(n)}\Big(|Z_2(n-t_1h(n),n)-r_{n2}z_2|>\Lambda_{n2}(t_1)|\hat{\mathbf{Z}}(n;t_0)=\hat{\mathbf{z}}_n\Big)&\\
\le \mathbf{E}\Big[|Z_2(n-t_1h(n),n)-r_{n2}z_2|^{\gamma}\,|\hat{\mathbf{Z}}(n;t_0)=\hat{\mathbf{z}}_n,\mathbf{Z}(n)\neq\mathbf{0}\Big]&\\
\le 2\mathbf{E}\left[\sum_{k=1}^{z_2}|\zeta_k(n;t_0,t_1)-r_{n2}|^{\gamma}\,|\hat{\mathbf{Z}}(n;t_0)=\hat{\mathbf{z}}_n,\mathbf{Z}(n)\neq\mathbf{0}\right]&\\
=2z_2\mathbf{E}_2\left[|Z_2((t_0-t_1)\,h(n),t_0h(n))-r_{n2}|^{\gamma}\,|\mathbf{Z}(t_0h(n))\neq\mathbf{0}\right]&\\
\le 2z_2\left(\frac{2^{\gamma}}{Q_{21}(t_0h(n))}\mathbf{E}_2\left[Z_2^{\gamma}((t_0-t_1)\,h(n),t_0h(n))\right]+2^{\gamma}r_{n2}^{\gamma}\right)&,
\end{aligned}$$

where in the last inequality we have applied, for $a,b\in\mathbf{R}_2$

$$|a-b|^{\gamma}\le\max(|2a|^{\gamma},|2b|^{\gamma})\le 2^{\gamma}(|a|+|b|). \tag{63}$$

Further, let

$$I_k\left(t_0,t_1;n\right),\quad k=1,2,...,Z_2((t_0-t_1)\,h(n))$$

be the indicator of the event that the kth particle belonging to generation $(t_0-t_1)\,h(n)$ has descendants in generation $t_0h(n)$ and $I_k\left(t_0,t_1;n\right)\overset{d}{=}I\left(t_0,t_1;n\right)$. Now, again by (63) and the Bahr-Esseen inequality

$$\begin{aligned}
&2^{-\gamma}\mathbf{E}_2\left[|Z_2((t_0-t_1)\,h(n),t_0h(n))|^{\gamma}\right]\\
&\quad=2^{-\gamma}\mathbf{E}_2\left[\left|\sum_{k=1}^{Z_2((t_0-t_1)h(n))}\left(I_k\left(t_0,t_1;n\right)-Q_2(t_1h(n))\right)+Z_2((t_0-t_1)\,h(n))Q_2(t_1h(n))\right|^{\gamma}\right]\\
&\quad\le\mathbf{E}_2\left[\left|\sum_{k=1}^{Z_2((t_0-t_1)h(n))}\left(I_k\left(t_0,t_1;n\right)-Q_2(t_1h(n))\right)\right|^{\gamma}\right]+\mathbf{E}_2\left[Z_2^{\gamma}((t_0-t_1)\,h(n))Q_2^{\gamma}(t_1h(n))\right]\\
&\qquad\le\mathbf{E}_2\left[Z_2((t_0-t_1)\,h(n))\right]\mathbf{E}_2\left[\left|\mathbf{1}_{\{Z_2(t_1h(n))>0\}}-Q_2(t_1h(n))\right|^{\gamma}\right]\\
&\qquad\qquad+Q_2^{\gamma}(t_1h(n))\mathbf{E}_2\left[Z_2^{\gamma}((t_0-t_1)\,h(n))\right].
\end{aligned}$$

Observe now that

$$\begin{aligned}
&\mathbf{E}\left[\left|\mathbf{1}_{\{Z_2(t_1h(n))>0\}}-Q_2(t_1h(n))\right|^{\gamma}\right]\\
&\qquad=Q_2(t_1h(n))\,|1-Q_2(t_1h(n))|^{\gamma}+\left(1-Q_2(t_1h(n))\right)Q_2^{\gamma}(t_1h(n))\le CQ_2(t_1h(n))
\end{aligned}$$

and by Lemma 11 in [18]

$$\mathbf{E}_2\left[Z_2^{\gamma}((t_0-t_1)\,h(n))\right]\le CQ_2^{1-\gamma}((t_0-t_1)\,h(n))\le CQ_2^{1-\gamma}(t_0h(n)).$$

As a result we get

$$\mathbf{E}_2\left[Z_2^{\gamma}((t_0-t_1)\,h(n),t_0h(n))\right]\le CQ_2(t_1h(n)).$$

This shows that, for each fixed pair $0<p_1<p_2<\infty$

$$\mathbf{P}^{(n)}\Big(|Z_2(n-t_1h(n),n)-r_{n2}z_2|>\Lambda_{n2}(t_1)|\hat{\mathbf{Z}}(n;t_0)=\hat{\mathbf{z}}_n\Big)\le Cz_2\Big(\frac{1}{\Lambda_n(t_1)}\Big)^{\gamma}\le C\Big(\frac{1}{\Lambda_n(p_1)}\Big)^{\gamma-1}$$

which goes to 0 as $n\to\infty$ uniformly in $(t_0,t_1)\in[p_1,p_2]$ and

$$z_2\le C\Lambda_{n2}(p_1)\quad\text{or}\quad\hat{z}_2=z_2/\Lambda_{n2}(t_1)\le C_1.$$

Further, for $\gamma \in (1, 1+\alpha_1)$ we have

$$\mathbf{P}^{(n)}\left(\left|Z_1^{(1)}(n - t_1h(n), n; t_0) - r_{n1}z_1\right| > \Lambda_{n1}(t_1) |\, \hat{\mathbf{Z}}(n; t_0) = \hat{\mathbf{z}}_n\right)$$

$$\leq \left(\frac{1}{\Lambda_{n1}(t_1)}\right)^{\gamma} \mathbf{E}_2\left[\left|Z_1^{(1)}(n - t_1h(n), n; t_0) - r_{n1}z_1\right|^{\gamma} |\, \hat{\mathbf{Z}}(n; t_0) = \hat{\mathbf{z}}_n, \mathbf{Z}(n) \neq \mathbf{0}\right].$$

We now use the representation

$$Z_1^{(1)}(n - t_1h(n), n; t_0) = \sum_{k=1}^{Z_1(n - t_0h(n), n)} \eta_k(n; t_0, t_1)$$

where $\eta_{1k}(n; t_0, t_1) \stackrel{d}{=} \eta(n; t_0, t_1)$ and $\eta(n; t_0, t_1)$ is the number of type 1 particles in the reduced process at time $n - t_1h(n)$ being descendants of a type 1 particle existing in the reduced process at time $n - t_0h(n)$. Observe that

$$\mathbf{E}\left[\eta^{\gamma}(n; t_0, t_1)\right] \leq \mathbf{E}_1\left[Z_1^{\gamma}(n - t_1h(n))\right] < \infty$$

for any $\gamma \in [1, 1+\alpha_1)$ in view of Condition (2) and Theorem 2.3.7 in [12]. Similarly to the previous estimates

$$\mathbf{E}\left[\sum_{k=1}^{Z_1(n - t_0h(n), n)} |\eta_k(n; t_0, t_1) - r_{n1}|^{\gamma} \,|\hat{\mathbf{Z}}(n; t_0) = \hat{\mathbf{z}}_n, \mathbf{Z}(n) \neq \mathbf{0}\right]$$

$$\leq 2z_1\mathbf{E}_1\left[\left|\eta(n; t_0, t_1) - \frac{Q_1(t_1h(n))}{Q_1(t_0h(n))}\right|^{\gamma} |Z_1(n - t_0h(n), n) = 1\right]$$

$$= 2z_1\mathbf{E}_1\left[\left|Z_1((t_0 - t_1)\, h(n), t_0h(n)) - \frac{Q_1(t_1h(n))}{Q_1(t_0h(n))}\right|^{\gamma} |Z_1(t_0h(n)) > 0\right] \leq Cz_1.$$

Hence, for each fixed pair $0 < p_1 < p_2 < \infty$ we get

$$\mathbf{P}^{(n)}\left(\left|Z_1^{(1)}(n - t_1h(n), n; t_0) - r_{n1}z_1\right| > \Lambda_{n1}(t_1) |\mathbf{Z}(n; t_0) = \mathbf{z}\right) \leq \left(\frac{1}{\Lambda_{n1}(t_1)}\right)^{\gamma} z_1 \to 0$$

as $n \to \infty$ uniformly in $(t_0, t_1) \in [\mathbf{p}_1, \mathbf{p}_2]$ and $z_1 \leq C\Lambda_{n1}^{\gamma}(p_1)$ or $\hat{z}_1 = z_1/\Lambda_{n1}^{\gamma}(t_1) \leq C_1$.

Finally, using the branching property, we get

$$\Lambda_{n1}(t_1)\mathbf{P}^{(n)}\left(Z_1^{(2)}(n - t_1h(n), n; t_0) > \Lambda_{n1}(t_1) |\, \hat{\mathbf{Z}}(n; t_0) = \hat{\mathbf{z}}_n\right)$$

$$\leq \mathbf{E}\left[Z_1^{(2)}(n - t_1h(n), n; t_0) |\, \hat{\mathbf{Z}}(n; t_0) = \hat{\mathbf{z}}_n, \mathbf{Z}(n) \neq \mathbf{0}\right]$$

$$= z_2\mathbf{E}_2\left[Z_1((t_0 - t_1)\, h(n); t_0h(n)) |\, \mathbf{Z}(t_0h(n)) \neq \mathbf{0}\right]$$

$$= z_2\frac{A_{21}\,(t_0 - t_1)\, h(n)Q_1(t_1h(n))}{Q_{21}(t_0h(n))} \leq C\frac{z_2}{\Lambda_{n1}(t_1)}.$$

Thus, for each fixed pair $0 < p_1 < p_2 < \infty$

$$\mathbf{P}^{(n)}\left(Z_1^{(2)}(n - t_1h(n), n; t_0) > \Lambda_{n1}(t_1) |\hat{\mathbf{Z}}(n; t_0) = \hat{\mathbf{z}}_n\right) \leq C\frac{z_2}{\Lambda_{n1}^2(t_1)} \to 0$$

as $n \to \infty$ uniformly in $(t_0, t_1) \in [p_1, p_2]$ and $\hat{z}_2 = z_2/\Lambda_{n1}(t_1) \leq C_1$.

The remaining part of the proof follows the line of proving Theorem 6(1) and is based on the one-dimensional convergence established in Theorem 3(2). We omit the details.

10. Most recent common ancestor

We present in this section the derivation of the laws of the MRCA. They are direct consequences of Theorems 4-6.

Point (1): It follows from Theorem 4 that, given Condition (8) particles of type 2 are always present in the limiting process. Thus, the MRCA should be of type 2. Since, given $\mathbf{Z}(n) \neq \mathbf{0}$ the event $\{Z_2(m,n)=1\}$ has a nonnegligible probability only if $\lim_{n\to\infty} mn^{-1} < 1$, to find the distribution of the MRCA in this situation it is sufficient to evaluate the derivative with respect to s_2 of the limiting probability generating function (14) at point $s_2 = 0$. This gives, as desired

$$\lim_{n\to\infty} \mathbf{P}_2\left(Z_2\left(an,n\right)=1|\mathbf{Z}(n)\neq\mathbf{0}\right) = \lim_{n\to\infty} \mathbf{P}_2\left(\beta_n \geq an|\mathbf{Z}(n)\neq\mathbf{0}\right) = 1-a.$$

Point (2): According to the offspring generating function of the limit process, given in (24), the law of the offspring vector (ξ_{21},ξ_{22}) produced by a type 2 individual satisfies:

$$\mathbf{P}(\xi_{21}=0,\xi_{22}\geq 2) = \frac{\alpha_2}{1+\alpha_2} \quad \text{and} \quad \mathbf{P}(\xi_{21}=1,\xi_{22}=0) = \frac{1}{1+\alpha_2}. \tag{64}$$

Hence there are two possibilities:

- Either the initial type 2 individual generates several type 2 individuals at the firth birth event. Then the time of the branching corresponds to the death of the MRCA.
- Or the initial type 2 individual gives birth to a type 1 individual at the first birth event. Then the process does not evolve anymore as the type 1 individuals are immortal and sterile. We will see that in this case the MRCA birth time is of order larger than $g^*(n)$.

From (64) we get that

$$\mathbf{P}(Y_1(t)+Y_2(t)=1 \quad \text{for any } t>0) = \frac{1}{1+\alpha_2}.$$

Moreover, from points (2)-(3) of Theorem 2, we see that for $g^*(n) \ll m \ll n$

$$\lim_{n\to\infty} \mathbf{P}_2(Z_1(m,n)=1|\mathbf{Z}(n)\neq 0) = \frac{1}{1+\alpha_2},$$

and for $a\in(0,1)$

$$\lim_{n\to\infty} \mathbf{P}_2(Z_1(an,n)=1|\mathbf{Z}(n)\neq 0) = \frac{1-a}{1+\alpha_2}.$$

As the process $Z_1(m,n)$ is non-decreasing with m, we deduce that for $g^*(n) \ll m \ll n$ and $a\in(0,1)$,

$$\lim_{n\to\infty} \mathbf{P}_2(m \ll \beta_n \leq an|\mathbf{Z}(n)\neq 0) = \frac{a}{1+\alpha_2}.$$

If, on the contrary the initial type 2 individual gives birth to several type 2 individuals, which happens with a probability $\alpha_2/(1+\alpha_2)$, the time of this first branching will be an exponential variable with parameter $(1+\alpha_2)/\alpha_2$. Converting this distribution in terms of the limit law for the MRCA yields

$$\lim_{n\to\infty} \mathbf{P}_2(\beta_n \leq tg^*(n)|\mathbf{Z}(n)\neq 0) = \frac{\alpha_2}{1+\alpha_2}(1-e^{-t(1+\alpha_2)/\alpha_2}).$$

This ends the proof of Theorem 7(2).

Point (3): To find the distribution of the MRCA birth time under Condition (10) we use the offspring generating functions of the process $\{\mathbf{W}(t), t\geq 0\}$, which are given in (27) and (28). Equation (28) implies the following properties for the law of the offspring vector (ξ_{21},ξ_{22}) produced by a type 2 individual of $\mathbf{W}(\cdot)$:

$$\mathbf{P}_2(\xi_{21}=1,\xi_{22}=0) = \frac{\sigma\alpha_2 A_{21}}{b((1+\alpha_2)b^{\alpha_2}-1)}, \ \mathbf{P}_2(\xi_{21}=0,\xi_{22}\geq 2) = \frac{\alpha_2 b^{\alpha_2}}{(1+\alpha_2)b^{\alpha_2}-1}. \tag{65}$$

Moreover, we see from the definition of $W_1(\cdot)$ and (27) that a type 1 individual dies after an exponentially distributed time with parameter 1 and gives birth to two type 1 individuals at least. Hence if we denote by T the time of the first branching there are two possibilities:

- Either the initial type 2 individual gives birth to several type 2 individuals. In this case T will be an exponential variable with parameter κ, where κ has been defined in (26). According to (65) this happens with probability
$$\frac{\alpha_2 b^{\alpha_2}}{(1+\alpha_2)b^{\alpha_2}-1} = \frac{b^{\alpha_2}}{\kappa}.$$
- Or the initial type 2 individual gives birth to one type 1 individual. Then this type 1 individual has an exponential distributed life-time with parameter 1. In this case, T will be the sum of an exponential variable with parameter κ and of an independent exponential variable with parameter 1. According to (65) this happens with probability
$$\frac{\sigma\alpha_2 A_{12}}{b((1+\alpha_2)b^{\alpha_2}-1)} = \frac{\sigma A_{21}}{b\kappa}.$$

Hence we get
$$\mathbf{P}(T \le t) = \frac{b^{\alpha_2}}{\kappa}(1-e^{-\kappa t}) + \frac{\sigma A_{12}}{b\kappa}\Big(1 + \frac{1}{\kappa-1}e^{-\kappa t} - \frac{\kappa}{\kappa-1}e^{-t}\Big).$$
By using several times the equality $b^{1+\alpha_2} - b = \sigma\alpha_2 A_{12}$ given in (19), we finally obtain
$$\mathbf{P}(T \le t) = 1 - \frac{1}{1+\alpha_2}e^{-t} - \frac{\alpha_2}{1+\alpha_2}e^{-t((1+\alpha_2)b^{\alpha_2}-1)/\alpha_2}.$$
Then by applying Theorem 6(1), we get for every $a \in (0,1)$:
$$\lim_{n\to\infty} \mathbf{P}(\beta_n \le an) = \mathbf{P}(T \le -\ln(1-a)).$$
This ends the proof of Theorem 7(3).

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